Optimized Impulses for Multicarrier Offset-QAM

Stephan Pfletschinger, Joachim Speidel
Institut für Nachrichtenübertragung
Universität Stuttgart, Pfaffenwaldring 47, D-70469 Stuttgart, Germany

Abstract – **The design of optimum pulses for digital multicarrier Offset-QAM (MC-OQAM) is presented. The impulses are designed as finite duration pulses with maximum signal energy in one subcarrier's frequency band. The intersymbol and intercarrier interference is minimized. The solution is obtained by expanding the impulses into finite-length discrete prolate spheroidal sequences. The design method, optimum impulses and simulation results are presented. With this technique, more than 95% of the signal energy can be concentrated in the frequency band of one subcarrier and the out-of-band spectral parts fall off very rapidly in comparison with rectangular pulses used by conventional OFDM systems. Thus, the presented multicarrier system provides much less spectral overlap between modulated subcarriers and is therefore much less susceptible against frequency-selective interference.**

I. Introduction

Multicarrier systems are nowadays widely used in wireless (DAB, DVB-T, radio LAN) as well as in wireline systems (xDSL). Its main advantage over single carrier systems are the immunity against impulsive noise and multipath fading and the absence of a complex equalizer.

The great majority of OFDM systems use FFT-processing with a cyclic prefix as a guard interval in order to combat intersymbol (ISI) and interchannel interference (ICI). This solution is distinguished by its implementational simplicity: modulation and filtering is done with FFT-processing which can be realized very efficiently and the cyclic prefix is merely a repetition of some symbols, but nevertheless it removes quite efficiently to a great extent the ISI and ICI. However, the drawbacks of this solution are the violation of the matched filter criterion at the receiver due to the cyclic prefix and the slowly decreasing spectra of the modulated subcarriers as the FFT-processing corresponds to a filtering with rectangular impulses. Therefore, the modulated subcarriers show considerable spectral overlap which leads to the effect that frequency-selective noise at one frequency not only affects the subchannel at this frequency but also – to a certain extent – the adjacent subchannels. Furthermore, the spectral efficiency is reduced by the insertion of the guard interval.

Ultimately, there has been an increasing interest in impulse shaping [1], [2], [3]. By designing proper impulses, the spectrum of the modulated subcarriers can be well localized to minimize spectral overlap and reduce sensitivity to frequency-selective interference.

II. System Model

Our optimization is based on the system model in Fig. 1 which is the baseband equivalent of a MC-OQAM system with N subcarriers. It can be considered as a straightforward extension of a single carrier OQAM.

The incoming symbols $X_\nu[k]$ arrive at symbol rate $1/T$ and are first upsampled with factor $M \geq N$, before their real and imaginary parts are filtered with the FIR filters $g[n]$ and $g[n-M/2]$, respectively. For even carrier indices ν, the real part is filtered with $g[n]$ and the imaginary part with $g[n-M/2]$, and vice versa. In other words, the real and the imaginary part are delayed alternately by half a symbol period. After filtering, the signals are frequency shifted by multiplication with $w^{-\nu \cdot n}$, where $w = \exp(-j2\pi/M)$. This corresponds to a modulation with $\exp(j\nu(2\pi/T)\cdot t)$ and sampling at $t = n \cdot T/M$. Thus, the carrier spacing is given by

$$\Delta\omega = \frac{2\pi}{T} \quad . \tag{1}$$

The sum of all subcarriers constitutes the output signal which may represent the baseband equivalent (complex envelope) of a bandpass signal or which may be made real by imposing proper conditions on the $X_\nu[k]$ like is done in DMT-modulation. An efficient implementation of the system in Fig. 1 using FFT-processing and polyphase filter banks is presented in [4].

For orthogonality, we require that for a unit impulse that is sent in one subchannel we receive a unit impulse on the same subchannel and zeros elsewhere:

For $X_l[k] = \delta[l-\nu]\cdot\delta[k]$

we receive $Y_\mu[k] = \delta[\nu-\mu]\cdot\delta[k]$, (2a)

where $\delta[k]$ is the Kronecker delta.

As the real and imaginary parts of the input sequences are processed differently, we need a similar condition for an imaginary unit impulse:

For $X_l[k] = j\cdot\delta[l-\nu]\cdot\delta[k]$

we receive $Y_\mu[k] = j\cdot\delta[\nu-\mu]\cdot\delta[k]$ (2b)

This condition (2a,b) is often referred to as extended Nyquist criterion because it defines ISI- and ICI-free transmission. It is also called orthogonality condition or condition for perfect reconstruction [5]. Considering the system in Fig. 1,

Global Telecommunications Conference, 2001. GLOBECOM '01. IEEE 25-29 November 2001, vol.1, pag 207 - 211

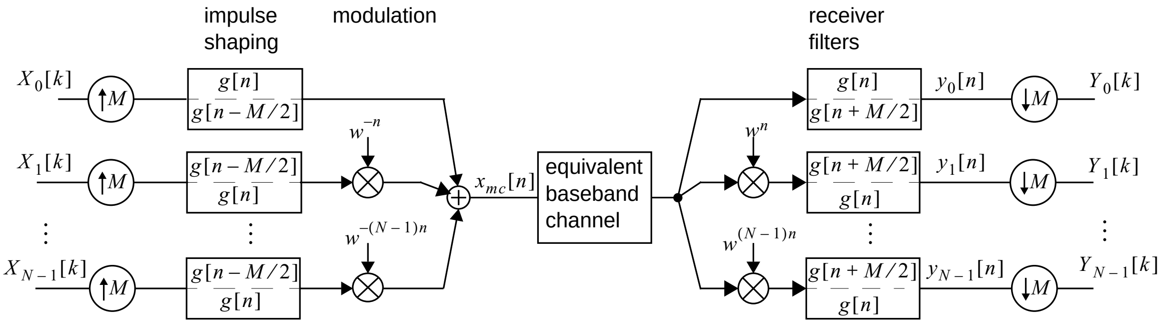

Fig. 1. Block diagram of a digital multicarrier offset-QAM system. The impulse shaping filters alternately delay the real part or the imaginary part of their input signals by half a symbol period.

(2a) leads with $d = \nu - \mu$ to the condition on the filter function:

$$\sum_{m=-\infty}^{\infty} g\left[kM - m - (\nu \mathrm{Mod} 2)\frac{M}{2}\right] \cdot \left\{ \cos\left(\frac{2\pi}{M}dm\right) g\left[m + (\mu \mathrm{Mod} 2)\frac{M}{2}\right] - j\sin\left(\frac{2\pi}{M}dm\right) \cdot g\left[m + ((\mu+1)\mathrm{Mod} 2)\frac{M}{2}\right] \right\} = \delta[k]\cdot\delta[d] \quad (3)$$

For practical implementations, the filter length, which is defined as LM, is limited and we assume for the following that the filter function is even:

$$g[n] = g[-n] \quad (4a)$$

$$g[n] = 0 \text{ for } |n| \geq LM/2 \quad (4b)$$

With these assumptions which are valid throughout this paper, it can be shown that (3) is already fulfilled for odd d and we can express the resulting orthogonality condition (extended Nyquist criterion) for even $d = 2\chi$ as:

$$r[k,\chi] = \sum_{n=-(L-k)M/2+1}^{(L-k)M/2-1} g\left[n + \frac{kM}{2}\right] \cdot g\left[n - \frac{kM}{2}\right] \cdot \cos\left(\frac{4\pi}{M}\chi n\right) = \delta[k]\cdot\delta[\chi] \quad (5)$$

with $\chi = 0, \ldots, N/2 - 1$, $k = 0, \ldots, L-1$

For (2b), the analogous procedure leads as well to the above condition, so we can consider (5) as the orthogonality condition for the filter function $g[n]$. In fact, (5) is a set of $LN/2$ equations while the impulse contains $LM/2$ independent values, namely $g[0], \ldots, g[LM/2-1]$. In order to obtain some degrees of freedom for optimization of the spectrum of $g[n]$ we have to choose $M > N$.

If we take the same steps for the analysis of a multicarrier system without offset, i.e. in the block diagram we replace the filters with $g[n]$ for both real and imaginary parts, we end up with an equation system similar to (5), but with LN equations. Therefore, the advantage of the Offset-QAM is obvious: the number of orthogonality conditions is reduced to the half.

For $M = N$ we would have the "critically sampled" or "maximum decimated" case [5], like usually realized in standard OFDM implementations. In an OFDM system with an N-point FFT and nominally N subcarriers, there are always much less than N carriers actually modulated (e.g. in DVB-T: $N = 8192$, 6817 used). For the further analysis we choose

$$M = 2N, \quad (6)$$

which gives us N degrees of freedom for optimization.

Please note that the spectral efficiency does not depend on the choice of M as the required bandwidth is always given by $\omega_B = N \cdot \Delta\omega$ and $\Delta\omega$ according to (1) does not depend on M.

III. Concentration of the Spectrum

A. Definition of the In-Band Energy

In order to achieve a good spectral concentration of the modulated subcarriers, it is desirable to put as much signal energy of $g[n]$ as possible inside the frequency band

$$|\omega| \leq \eta \cdot \frac{\Delta\omega}{2} \quad (7)$$

where η denotes a parameter that defines the width of the considered frequency band. An ideal solution is the rectangular lowpass with a time function of type $\mathrm{sinc}(t) = \sin(t)/t$, which cannot be implemented due to its infinite length.

In [1] an optimization was developed for an analog model and the parameter $\eta = 2$, while in this paper we focus on a discrete-time implementation and choose the frequency band as narrow as one subcarrier's interval, i.e. $\eta = 1$.

The signal energy inside the frequency band (7) is given by

$$E_\eta = \frac{T_A}{2\pi} \int_{-\eta\Delta\omega/2}^{\eta\Delta\omega/2} |G(e^{j\omega T_A})|^2 d\omega = \frac{\eta}{2N} \sum_{n=-LN+1}^{LN-1} g[n] \sum_{m=-LN+1}^{LN-1} g[m]\,\mathrm{sinc}\left(\frac{\eta\pi}{2N}(n-m)\right) \quad (8)$$

where $G(z)$ denotes the double-sided z-transform of $g[n]$, and $T_A = T/M$ is the sampling period of $g[n]$. The second term is obtained by applying Parseval's theorem.

B. Discrete Prolate Spheroidal Sequences

For the maximization of the signal energy E_η we expand the impulse $g[n]$ into a series of indexlimited orthogonal sequences. In principle, each set of orthogonal sequences could be used for the expansion, but there exists one set of sequences that is especially well suited for this purpose: the discrete prolate spheroidal sequences (dpss), also named Slepian sequences [6]. They are distinguished in that they are the set of indexlimited orthogonal sequences which have maximum spectral energy in the frequency range (7). The continuous-time counterparts of the dpss have also been considered for optimal impulse shaping in single carrier systems, [7]. The dpss $v_l[n]$ are defined by

$$2W \sum_{m=0}^{N_p-1} \mathrm{sinc}(2\pi W(n-m)) v_l[m] = \lambda_l v_l[n] \qquad (9)$$

with $n \in Z$, $l \in \{0, \ldots, N_p - 1\}$, and depend on the parameters N_p and W. The λ_l are the corresponding eigenvalues, for which holds:

$$1 \geq \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{N_p-1} > 0$$

The dpss are orthogonal on both the intervals $0, \ldots, N_p - 1$ and on $-\infty, \ldots, \infty$:

$$\sum_{n=0}^{N_p-1} v_i[n] \cdot v_j[n] = \lambda_i \sum_{n=-\infty}^{\infty} v_i[n] \cdot v_j[n] = \delta[i-j] \qquad (10)$$

As we only deal with time limited impulses, we will make use of the first property only.

The dpss show symmetry with respect to $n = (N_p - 1)/2$: the sequences with even index l show an even symmetry while the sequences with odd index exhibit an odd symmetry:

$$v_l[n] = (-1)^l \cdot v_l[N_p - 1 - n]$$

We can now expand the impulse $g[n]$ into a series with the dpss as basis functions, taking into account the symmetry properties of both sequences.

$$g_1[n] = g[n - LN] = \sum_{i=0}^{LN} a_i v_{2i}[n] \qquad (11)$$

with $\quad a_i = \sum_{n=0}^{2LN} g_1[n] v_{2i}[n] \quad$ and $\quad N_p = 2LN + 1$

Putting (11) into (8), the signal energy as a function of the expansion coefficients a_i becomes

$$E_\eta = \frac{\eta}{2N} \sum_{i=0}^{LN} a_i \sum_{j=0}^{LN} a_j \sum_{n=0}^{2LN} v_{2i}[n] \sum_{m=0}^{2LN} v_{2j}[m] \, \mathrm{sinc}\left(\frac{\eta\pi}{2N}(n-m)\right)$$

Using the definition (9) of the dpss with $W = \eta/(4N)$ and the orthogonality relation (10) we obtain

$$E_\eta = \sum_{i=0}^{LN} a_i^2 \cdot \lambda_{2i} \qquad (12)$$

From (12) and (11) we conclude that the eigenvalue λ_l represents the signal energy of the corresponding dpss inside the frequency band (7).

From Table I we find that for $l > L$ the eigenvalues decrease rapidly, i.e. the first dpss up to the index $l = L$ concentrate their signal energy inside the subcarrier frequency band.

IV. Numerical Solution and Simulation Results

The orthogonality condition (5) can be expressed by means of (11) in the expansion coefficients a_i. Thus, together with (12) we can formulate a nonlinear optimization problem for the expansion coefficients a_i. The numerical solution of this problem has been studied and has shown to be feasible but becomes computationally very expensive for many carriers.

To simplify the computation of the expansion coefficients, we make use of the energy concentration of the first sequences $v_l[n]$ and approximate the desired function $g_1[n] = g[n - LN]$ by only few sequences:

$$g_1[n] = \sum_{i=0}^{N_a} a_i \cdot v_{2i}[n] \qquad (13)$$

with $N_a \leq LN$. With (4a,b), (6), (11) we can express the orthogonality relation (5) as

$$r[k, \chi] = \sum_{n=kN}^{(2L-k)N} g_1[n + kN] \cdot g_1[n - kN] \cdot \cos\left(\frac{2\pi}{N}\chi n\right) = \delta[k, \chi] \qquad (14)$$

From this, it follows the orthogonality condition for the expansion coefficients:

$$r[k, \chi] = \sum_{i=0}^{N_a} a_i \sum_{j=0}^{N_a} a_j \underbrace{\sum_{n=kN}^{(2L-k)N} v_{2i}[n-kN] \cdot v_{2j}[n+kN] \cdot \cos\left(\frac{2\pi}{N}\chi n\right)}_{A[i,j,p]}$$

TABLE I

EIGENVALUES λ_l FOR EVEN INDICES. $N_p = 2LN + 1$, $W = \eta/(4N)$

	eigenvalues λ_l		
index l	$L = 4$	$L = 6$	$L = 8$
0	0.99995	1	1
2	0.96213	0.99974	1
4	0.28724	0.9494	0.99946
6	0.0038201	0.30166	0.94021
8	8.4332e-06	0.0064271	0.3108
10	6.7196e-09	2.9067e-05	0.0088072
>10	$<10^{-11}$	$<10^{-7}$	$<10^{-4}$

where $p = k \cdot \frac{N}{2} + \chi$, $k = p \text{ Div} \frac{N}{2}$, $\chi = p \text{ Mod} \frac{N}{2}$

So, this procedure results in a deviation of the Nyquist criterion. We define the quadratic error

$$J = \sum_{k=0}^{L-1} \sum_{\chi=0}^{N/2-1} (r[k,\chi] - \delta[k,\chi])^2 \qquad (15)$$

which represents the deviation from the extended Nyquist criterion according to (5). J has to be minimized with respect to the a_i, $(i = 0, \ldots, N_a)$. Separating the terms with $k = \chi = 0$ we obtain

$$J = e + (r[0,0] - 1)^2,$$

where e contains the ISI and ICI energy. The second term shows the deviation of $r[0,0]$ from the ideal value 1. Making this deviation to zero results in

$$r[0,0] = 1 \quad \Rightarrow \sum_{i=0}^{N_a} a_i^2 = 1 \qquad (16)$$

As can be shown, this is equivalent to the normalization of the signal energy of $g[n]$. With (16) one coefficient a_i is settled and we get

$$J = e = \sum_{p=1}^{LN/2-1} c_p^2 \qquad (17)$$

with $\quad c_p = \sum_{i=0}^{N_a} a_i \sum_{j=0}^{N_a} a_j \cdot A[i,j,p] = \boldsymbol{a} \boldsymbol{A}_p \boldsymbol{a}^T$

A. Solution for Expansion into three Sequences

For a filter length parameter $L = 4$, the first three sequences with even index already concentrate nearly all spectral energy in the subcarrier frequency band, as can be seen in

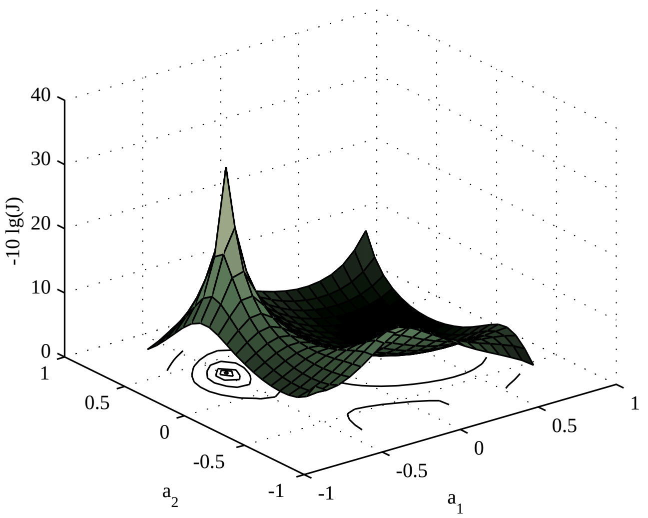

Fig. 2. Squared error as a function of the expansion coefficients a_1 and a_2. The function is displayed in negative logarithmic form in order to make the minimum more visible. The function was calculated with the parameters $N = 256$, $L = 4$.

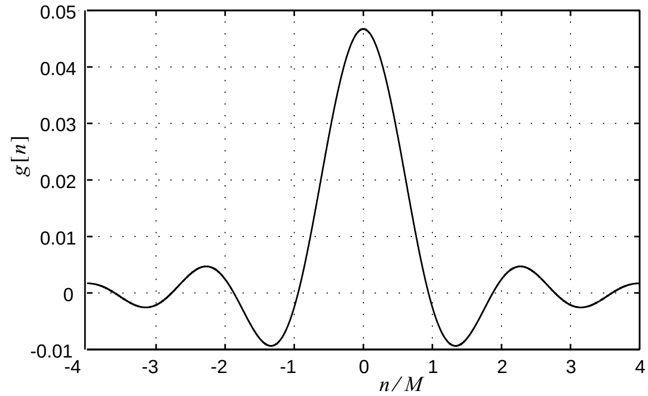

Fig. 3. Optimized impulse for $N = 256$, $L = 8$, $N_a = 6$.

Table I . Thus, expansion according to (13) with $N_a = 2$ guarantees already a good energy concentration. For this case, the minimization for the squared error (17) is quite illustrative: From (16) we get

$$a_0 = \sqrt{1 - a_1^2 - a_2^2}$$

which we can substitute in (17). Then, the squared error J only depends on a_1 and a_2 and can be visualized in a diagram, like is done in Fig. 2.

The search for the minimum produced a squared error of $J = 2.24 \cdot 10^{-4}$, or -36.5 dB. The signal energy located inside the subcarrier frequency band is $E_1 = 91.92\,\%$.

B. General Solution

For an expansion into more than only three Slepian sequences, the solution is obtained by minimizing the object function (17) with respect to the constraint (16). This can be easily done with standard numerical routines which can be seconded by providing analytical expressions for the gradient of object function and constraint. In Fig. 3, the optimized impulse for $N = 256$ carriers and $L = 8$ is presented.

Table II compares the achieved signal energy in the given frequency band of the optimized impulses with some other well-known impulse shapes. Also shown is the interference

TABLE II

SIGNAL ENERGY E_1 IN THE FREQUENCY BAND $|\omega| < \Delta\omega/2$ FOR DIFFERENT PULSE SHAPES. ALSO SHOWN IS THE POWER OF THE INTERSYMBOL (P_{ISI}) AND INTERCARRIER INTERFERENCE (P_{ICI}), RESPECTIVELY.

impulse shape	E_1	P_{ISI}	P_{ICI}
optimized impulse	95.78%	-50.77 dB	-52.66 dB
square root raised cosine	81.82%	-60.64 dB	-62.92 dB
Gauss-impulse	92.37%	-10.63 dB	-51.24 dB
rectangular impulse	77.29%	–	–[a]

a. ISI and ICI is eliminated by the use of a guard interval

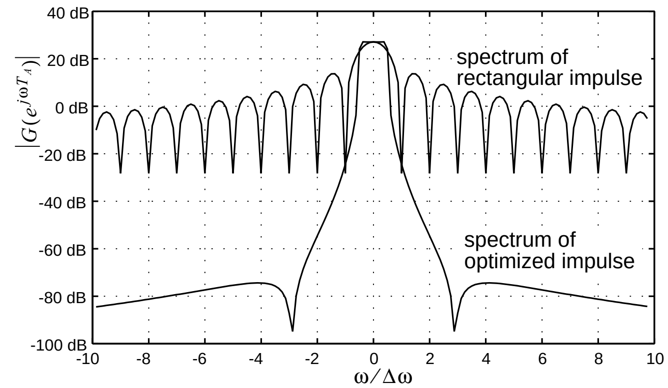

Fig. 4. Magnitude of the spectrum of the optimized impulse in comparison with the spectrum of the rectangular impulse used in conventional OFDM.

power due to ISI and ICI which is strongly related to the squared error (17). The square-root-raised-cosine impulse does not achieve the good energy concentration of the optimized impulse, but its interference power is slightly lower, as the untruncated impulse ideally fulfills the Nyquist criterion (2a,b). The Gauss-impulse $g(t) = \exp(-\pi(t/T)^2)$ has some interesting properties like its invariance with respect to the Fourier transform and the very fast decay of its spectrum. However, it does not fulfill the Nyquist theorem and substantial ISI occurs which has to be mitigated by powerful equalization techniques [3]. The rectangular impulse shows the poorest energy concentration, but its strength lies in the simple elimination of interference by the use of a guard interval.

The spectrum of the optimized impulse in Fig. 4 illustrates the good energy concentration compared to the rectangular pulse used in conventional OFDM. Around the carrier frequency a nearly flat spectrum is achieved while the out-of-band spectral parts fall off much more rapidly than for normal OFDM. This is the reason why the system with optimized impulses is much less susceptible to narrow-band interference.

C. Simulation Results

To investigate the susceptibility of the system against interference, the bit error rate (BER) is simulated and compared to standard OFDM. We assume a flat channel with a constant sinusoidal interferer at the frequency $\omega = 40.2\Delta\omega$ and an SNR of -5 dB. The BER on each subchannel are shown in Fig. 5. With our optimized MC-OQAM scheme, the interferer affects, as expected, only very few channels in the region of $\omega = 40\Delta\omega$. For all other subchannels the BER is zero. In contrast, the OFDM subchannels from $24\Delta\omega$ to $56\Delta\omega$ are heavily degraded. The total BER over all subchannels is $P_{MC-OQAM} = 1.3 \cdot 10^{-5}$ and $P_{OFDM} = 3.6 \cdot 10^{-2}$, respectively. For time-invariant channels, like with ADSL this advantage can be enhanced by using adaptive bitloading schemes which reduce the bitrate on the noise-affected carriers.

Simulations with AWGN showed identical results for both

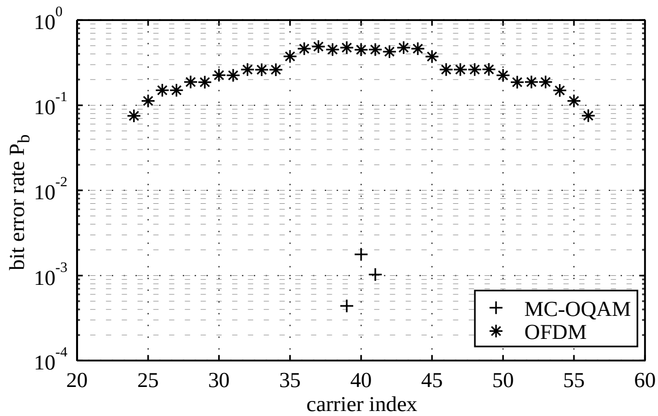

Fig. 5. Bit error rates on each subchannel in the presence of a sinusoidal interferer at $40.2\Delta\omega$. $N = 256$ subcarriers.

systems, the resulting bit error rate was equal to the theoretical value for QAM.

V. CONCLUSION

A computational efficient procedure for designing impulses for digital MC-OQAM based on expansion into discrete prolate spheroidal sequences has been presented. The impulses maximize the signal energy around the carrier frequencies, resulting in minimal spectral overlap between the adjacent subchannels. The out-of-band spectrum decays much faster than for the rectangular impulse which is used in standard OFDM and the out-of-band attenuation is improved by more than 20 dB. This is of great advantage if frequency-selective noise is present on the channel or for multiuser applications where spectral overlap has to be kept to a minimum to avoid heavy multi-user interference. Simulations showed that for narrow-band interference the bit error rate is reduced enormously.

REFERENCES

[1] A. Vahlin and N. Holte, "Optimal finite duration pulses for OFDM," *IEEE Trans. Comm.*, vol 44, no. 1, pp 10-14, Jan. 1996

[2] B. Le Floch, M. Alard, and C. Berrou, "Coded orthogonal frequency division multiplex" *Proc. IEEE*, vol. 83, no. 6, pp. 982-996, June 1995

[3] K. Matheus: *Generalized Coherent Multicarrier Systems for Mobile Communications*. Dr.-Ing. Dissertation, Aachen: Shaker, 1998

[4] N. J. Fliege, "Orthogonal multiple carrier data transmission," *European Transactions on Telecommunications*, vol. 3, no. 3, pp. 255-264, May-June 1992

[5] G. Cherubini, E. Eleftheriou, S. Ölçer, and J. Cioffi, "Filter bank modulation techniques for very high-speed digital subscriber lines," *IEEE Comm. Mag.*, pp. 98-104, May 2000

[6] D. Slepian, "Prolate spheroidal wave functions, fourier analysis, and uncertainty — V: the discrete case," *Bell Syst. Tech. J.*, vol. 57, no. 5, pp. 1371-1430, May-June 1978

[7] P. H. Halpern, "Optimum finite duration Nyquist signals," *IEEE Trans. Comm.*, vol. COM-27, no. 6, pp. 884-888, June 1979