Mutual Information of MIMO Channels in Correlated Rayleigh Fading Environments - a General Solution

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Abstract—We present a novel approach on the calculation of the moment generating function of mutual information of MIMO channels with correlated Rayleigh fading. For the first time, a concise mathematical formulation of the moment generating function is given in terms of a hypergeometric function of matrix arguments. In contrast to existing literature, our approach is not based on eigenvalue probability density functions but uses a direct integration technique. In principle, via the moment generating function it is possible to calculate exact, i.e. non-asymptotic moments, including e.g. ergodic capacity, for arbitrary array sizes and arbitrary correlation properties at receiver as well as transmitter, thus unifying and completing existing partial solutions for special propagation scenarios. Monte-Carlo simulations of ergodic capacity verify the accuracy of the analysis.

I. Introduction

In his seminal paper [1] Telatar calculated the ergodic capacity of a MIMO link with uncorrelated Rayleigh fading in an additive white Gaussian noise (AWGN) environment in terms of a single integral by integrating over the eigenvalue probability density function (PDF) of certain complex Wishart matrices, thereby predicting enormous capacity gains by combined spatial processing at transmitter and receiver and thus initiating immense research activities in this area. Later, Foschini and Gans presented numerical results and bounds on i.i.d. Rayleigh MIMO ergodic and outage capacity in their fundamental work [2]. Other bounds on the ergodic capacity of i.i.d. and correlated Rayleigh fading MIMO channels were given in [3][4][5][6][20]. Asymptotic results for large antenna arrays can be found in [7] and [8], where empirical eigenvalue PDFs of certain large dimensional random matrices are used.

By integrating over the eigenvalue PDF of an i.i.d. Wishart matrix, in [9] the moment generating function (MGF) of mutual information of an i.i.d. Rayleigh channel is derived. A similar MGF approach is taken in [10], where the authors present results for various propagation scenarios including e.g. i.i.d. and one-side correlated Rayleigh fading, as well as Ricean fading. Again, a mathematically challenging integration over the eigenvalue PDF of certain (non-central) Wishart matrices is necessary, prohibiting a general solution for Rayleigh fading with both receive and transmit correlation.

In this paper, we outline an eigenvalue PDF free approach for calculating the MGF of mutual information of correlated flat Rayleigh fading MIMO channels. The novel approach unifies all existing partial solutions and allows for a concise mathematical notation of the MGF in terms of hypergeometric functions of matrix arguments. Furthermore, it extends existing solutions such that it covers the case of fully correlated channels with both transmit as well as receive correlation.

As an example, in this paper the MGF is used to derive the exact ergodic capacity (EC) of a MIMO channel with full channel state information at the RX and no channel state information at the TX. The EC analysis is verified by Monte-Carlo simulations at the end of the paper. However, we note that another application is e.g. the calculation of Gamma or Gaussian approximations for the PDF of mutual information for the determination of outage capacities. On the other hand, numerical algorithms can be deployed for inverting the MGF, i.e. determining the PDF [9].

II. Signal and Channel Model

We consider a flat fading MIMO link modeled by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{n}\,, \tag{1}$$

where $\mathbf{s}$ is the T×1 TX symbol vector, $\mathbf{H}$ is the R×T MIMO channel matrix with correlated Rayleigh fading elements, $\mathbf{n}$ is the R×1 noise vector, and $\mathbf{y}$ is the R×1 receive vector. By R we denote the number of RX antennas and T is the number of TX antennas. In the following we assume additive Gaussian noise, where the noise covariance matrix is given by $\boldsymbol{R}_{nn} = N_0 \cdot \tilde{\boldsymbol{R}}_{nn}$. The signal covariance matrix is given by the analogue expression $\boldsymbol{R}_{ss} = E_s \cdot \tilde{\boldsymbol{R}}_{ss}$.

Using a widely accepted channel model [11], the correlated MIMO channel can be described by the matrix product

$$\boldsymbol{H} = \boldsymbol{A}^H \boldsymbol{H}_w \boldsymbol{B}\,, \tag{2}$$

where $\mathbf{H}_w$ is a R×T matrix of complex i.i.d. Gaussian variables of unity variance and

$$\boldsymbol{A}\boldsymbol{A}^H = \boldsymbol{R}_{RX} \qquad \boldsymbol{B}\boldsymbol{B}^H = \boldsymbol{R}_{TX}\,, \tag{3}$$

where $\mathbf{R}_{RX}$ and $\mathbf{R}_{TX}$ is the long-term stable (normalized) receive and transmit correlation matrix, respectively.

In the remainder of the paper, by $\mathbf{I}_n$ we denote an identity matrix of size n×n (the index can be omitted, if the size of the matrix is clear from the context), diag($x_1,\ldots x_n$) is a diagonal matrix with elements $x_1,\ldots x_n$, $|\boldsymbol{X}|$ is the determinant of the quadratic matrix $\mathbf{X}$, eig($\mathbf{X}$) returns a diagonal matrix of eigenvalues of $\mathbf{X}$, $\mathbf{X}^H$ means Hermitian, x≅means 'random variable (RV) x is statistically equivalent to', and $E_x[\ ]$ denotes expected value with respect to RV x.

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III. MIMO Mutual Information

A. General expressions

It is well known [1] that the mutual information $I(\boldsymbol{s}, \boldsymbol{y})$ between input vector **s** and output **y** of the MIMO link according to (1) is given by

$$I(\boldsymbol{s}, \boldsymbol{y}) = \log_2\left|\boldsymbol{I} + \gamma\tilde{\boldsymbol{R}}_{ss}\boldsymbol{H}^H\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{H}\right| \quad (4)$$

with the standard mean SNR per transmit symbol definition $\gamma = E_s/N_0$. Plugging the channel model (2) with Kronecker product covariance structure in (4), we find

$$I(\boldsymbol{s}, \boldsymbol{y}) = \log_2\left|\boldsymbol{I} + \gamma \cdot \tilde{\boldsymbol{R}}_{ss}\boldsymbol{B}^H\boldsymbol{H}_w^H\boldsymbol{A}\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{A}^H\boldsymbol{H}_w\boldsymbol{B}\right|. \quad (5)$$

In the following, we reduce (5) to a concise equivalent formulation that allows for a unified analysis of correlated MIMO systems. At this point, we emphasize that we assume full rank channel correlation and signal and noise covariance matrices in this paper. An extension is straightforward but would unnecessarily complicate notation, thus detracting from the main problems. By noticing that the distribution of the i.i.d. complex Gaussian distributed R×T matrix $\mathbf{H}_w$ is invariant to left- or right multiplications with unitary matrices **U** and **V**, i.e.

$$\boldsymbol{U}\boldsymbol{H}_w\boldsymbol{V} \cong \boldsymbol{H}_w, \quad (6)$$

we find after some simplifications

$$I(\boldsymbol{s}, \boldsymbol{y}) \cong \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{S}\boldsymbol{H}_w^H\boldsymbol{O}\boldsymbol{H}_w\right|. \quad (7)$$

In (7) we have introduced the R×R diagonal matrix of eigenvalues associated to the receive side

$$\boldsymbol{O} = \mathrm{eig}(\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{R}_{RX}) = \mathrm{diag}(o_1, \ldots, o_R), \quad (8)$$

which comprises the effects of receive fading correlation and colored additive Gaussian noise. Finally, $\boldsymbol{S}$ is a diagonal matrix with

$$\boldsymbol{S} = \mathrm{eig}(\tilde{\boldsymbol{R}}_{ss}\boldsymbol{R}_{TX}) = \mathrm{diag}(s_1, \ldots, s_T), \quad (9)$$

which takes into account fading correlation at the transmit antenna array and the signal covariance matrix. Note that we can alternatively formulate

$$I(s, y) \cong \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{S}\boldsymbol{H}_w^H\boldsymbol{O}\boldsymbol{H}_w\right| = \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{O}\boldsymbol{H}_w\boldsymbol{S}\boldsymbol{H}_w^H\right|. \quad (10)$$

We rewrite (10) such that the matrix argument $\boldsymbol{S}\boldsymbol{H}_w^H\boldsymbol{O}\boldsymbol{H}_w$ or $\boldsymbol{O}\boldsymbol{H}_w\boldsymbol{S}\boldsymbol{H}_w^H$, respectively, of the determinant is of full rank, thereby simplifying the subsequent analysis. To this end, we define

$$\mu \equiv \min(R, T) \qquad \nu \equiv \max(R, T), \quad (11)$$

the μ×μ diagonal matrix

$$\Sigma \equiv \begin{cases} \boldsymbol{S} & R \geq T \\ \boldsymbol{O} & T > R \end{cases} \qquad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_\mu), \quad (12)$$

the ν×ν diagonal matrix

$$\Omega \equiv \begin{cases} \boldsymbol{O} & R \geq T \\ \boldsymbol{S} & T > R \end{cases} \qquad \Omega = \mathrm{diag}(\omega_1, \ldots, \omega_\nu), \quad (13)$$

and the ν×μ matrix of i.i.d. complex Gaussian entries **G**. With above definitions we can introduce a unifying expression for MIMO mutual information, which will serve as a basis for all following derivations

$$I(\boldsymbol{s}, \boldsymbol{y}) \cong \log_2\left|\boldsymbol{I} + \gamma \cdot \Sigma\boldsymbol{G}^H\Omega\boldsymbol{G}\right|. \quad (14)$$

Furthermore, it turns out later that it is advantageous to consider $\nu \times \nu$ MIMO systems. To this end, we can artificially set (with ε_k taking on the value 0 in the limit for k=0, ..., $\nu - \mu - 1$)

$$\tilde{\Sigma}(\varepsilon) = \mathrm{diag}\Big(\sigma_1, \ldots, \sigma_\mu, \underbrace{\varepsilon_{\nu-\mu-1}, \ldots, \varepsilon_0}_{(\nu-\mu)\ \text{times}}\Big) = \mathrm{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_\nu). \quad (15)$$

Furthermore, we can introduce an enlarged $\nu \times \nu$ matrix $\tilde{\boldsymbol{G}}$ of i.i.d. complex Gaussian elements. Without loss of generality, we can thus write from (10) together with (15)

$$\begin{aligned} I(\boldsymbol{s}, \boldsymbol{y}) &\cong \lim_{\varepsilon \to 0} \log_2\left|\boldsymbol{I} + \Omega\tilde{\boldsymbol{G}}^H\tilde{\Sigma}(\varepsilon)\tilde{\boldsymbol{G}}\right| \\ &= \lim_{\varepsilon \to 0} \log_2\left|\boldsymbol{I} + \boldsymbol{Q}\right| = \lim_{\varepsilon \to 0} \log_2\left|\boldsymbol{I} + \Lambda\right| \end{aligned}, \quad (16)$$

which means that we can in the general case consider $\nu \times \nu$ MIMO systems with a subsequent limiting process. In (16), we have introduced **Q,** which is a $\nu \times \nu$ complex random matrix quadratic form with matrix of eigenvalues Λ.

B. Distribution of a complex generalized matrix quadratic form

The distribution of $\boldsymbol{Q}$ in (16) has been derived in [13]

$$p(\boldsymbol{Q}) = \frac{e^{-\mathrm{tr}(-q^{-1}\tilde{\Sigma}(\varepsilon)^{-1}\boldsymbol{Q})}}{\tilde{\Gamma}_\nu(\nu) \cdot \left|\tilde{\Sigma}(\varepsilon)\right|^\nu \cdot |\Omega|^\nu} \cdot {}_0\tilde{\boldsymbol{F}}_0^{(\nu)}(\boldsymbol{I}_\nu - q\Omega^{-1}; -q^{-1}\tilde{\Sigma}(\varepsilon)^{-1}\boldsymbol{Q}) \cdot (d\boldsymbol{Q}), \quad (17)$$

where q is an arbitrary scalar constant. In (17) we use the function ${}_0\tilde{F}_0^{(n)}$, which is a hypergeometric function of 2 matrix arguments, where n×n is the maximum dimension of the matrix arguments and in general we have the following series definition with two matrices **X** and **Y** ([15], equation (88))

$${}_p\tilde{\boldsymbol{F}}_q^{(n)}(a_1, \ldots, a_p; b_1, \ldots, b_q; \boldsymbol{X}, \boldsymbol{Y}) = \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[a_1]_\kappa \cdot \ldots \cdot [a_p]_\kappa}{[b_1]_\kappa \cdot \ldots \cdot [b_q]_\kappa} \cdot \frac{\tilde{C}_\kappa(\boldsymbol{X}) \cdot \tilde{C}_\kappa(\boldsymbol{Y})}{\tilde{C}_\kappa(\boldsymbol{I}_n) \cdot k!}. \quad (18)$$

In (18), $\kappa = (k_1, k_2, \ldots, k_n)$ is a partition of k into not more then n parts with $k_1 \geq k_2 \geq \ldots \geq k_n \geq 0$ and $k_1 + k_2 + \ldots + k_n = k$, and the complex multivariate hypergeometric coefficient reads

$$[a]_\kappa = \prod_{i=1}^{n}[a - i + 1]_{k_i} \quad (19)$$

with Pochhammer's symbol

$$[a]_k = a \cdot (a+1) \cdot \ldots \cdot (a+k-1) \qquad [a]_0 = 1. \quad (20)$$

The notation $\tilde{C}_\kappa(X)$ denotes a zonal polynomial defined in [15] and is a symmetric polynomial of degree k in the eigenvalues of matrix **X** and the interested reader is referred e.g. to [15] for greater details. Furthermore, the complex multivariate Gamma function is defined by (cf. [15], equation (83))

$$\tilde{\Gamma}_m(r) = \pi^{\frac{1}{2}m(m-1)} \cdot \prod_{i=1}^{m} \Gamma(r-i+1). \qquad (21)$$

We note that eigenvalue PDFs are available for the random matrix $\boldsymbol{Q}$ with PDF defined in (17). However, the joint eigenvalue PDF so far can only be given in terms of infinite sums of zonal polynomials for the case of channel correlation at transmitter as well as receiver ($\Sigma(\varepsilon) \neq \boldsymbol{I}$, $\Omega \neq \boldsymbol{I}$), thus complicating its practical use and so far prohibiting a general eigenvalue based capacity analysis for arbitrarily correlated channels.

*C. Calculation of the MGF of I(**s**,**y**)*

A MGF uniquely defines a probability distribution, i.e. once we can find the MGF of mutual information, we can determine all moments, including the practically important first moment, which is also known as ergodic capacity in MIMO literature for the case of an uninformed transmitter and full channel state information at the receiver.

To this end, we first focus on the artificial $\nu \times \nu$ system and have to calculate (after simple manipulations) the expected value with respect to $\boldsymbol{Q}$ (from (16))

$$\tilde{M}(s) = \mathrm{E}_{\boldsymbol{Q}}[e^{sI(\boldsymbol{s},\boldsymbol{y})}] = \mathrm{E}_{\boldsymbol{Q}}\left[|\boldsymbol{I}+\boldsymbol{Q}|^{\frac{s}{\ln 2}}\right], \qquad (22)$$

where $\tilde{M}(s)$ is the MGF of MIMO channel mutual information for the $\nu \times \nu$ MIMO system. The standard approach, e.g. applied in [9] to find the MGF, is to equivalently integrate over the eigenvalue PDF p(Λ) of $\boldsymbol{Q}$

$$\tilde{M}(s) = \mathrm{E}_{\Lambda}\left[|\boldsymbol{I}+\boldsymbol{Q}|^{\frac{s}{\ln 2}}\right] = \int_{\Lambda} |\boldsymbol{I}+\Lambda|^{\frac{s}{\ln 2}} \cdot p(\Lambda) d\Lambda. \qquad (23)$$

This integral has also been used in [10] for calculating the first and second moment of capacity for Ricean MIMO channels and Rayleigh channels with one-sided fading correlation.

IV. Novel Approach For MGF

Due to the problems with the eigenvalue based approach, in this section we introduce a novel direct integration technique for deriving the mutual information MGF.

A. Concise matrix notation

We start with the main theorem of this work.

Theorem 1. The MGF $\tilde{M}(s)$ of the MIMO channel mutual information according to (16) is given by

$$\tilde{M}(s) = {}_2\tilde{\boldsymbol{F}}_0^{(\nu)}\left(-\frac{s}{\ln 2}, \nu; -\tilde{\Sigma}(\varepsilon), \Omega\right). \qquad (24)$$

Proof: We directly focus on solving the integral (see (22))

$$\tilde{M}(s) = \int_{\boldsymbol{Q}} |\boldsymbol{I}+\boldsymbol{Q}|^{\frac{s}{\ln 2}} \cdot p(\boldsymbol{Q}) d\boldsymbol{Q}. \qquad (25)$$

The key for solving (25) is given by the following property of the hypergeometric function of matrix argument (cf. [15], equation (90)) for the n×n matrix **X**

$${}_1\tilde{\boldsymbol{F}}_0^{(n)}(a;\boldsymbol{X}) = |\boldsymbol{I}-\boldsymbol{X}|^{-a}, \qquad (26)$$

which is the matrix analog to the well-known scalar binomial series

$${}_1F_0(a;x) = (1-x)^{-a}, \qquad (27)$$

where ${}_1F_0$ is the scalar hypergeometric function. Application of (26) to (25) leads to

$$\tilde{M}(s) = \mathrm{E}_{\boldsymbol{Q}}\left[{}_1\tilde{\boldsymbol{F}}_0^{(\nu)}\left(-\frac{s}{\ln 2};-\boldsymbol{Q}\right)\right]. \qquad (28)$$

By [13], equation (58) we have for a Hermitian matrix **M**

$$E_{\boldsymbol{Q}}[\tilde{C}_\kappa(\boldsymbol{MQ})] = \frac{\tilde{C}_\kappa(\boldsymbol{M}\tilde{\Sigma}(\varepsilon)) \cdot \tilde{C}_\kappa(\Omega)}{\tilde{C}_\kappa(\boldsymbol{I})} \cdot [\nu]_\kappa. \qquad (29)$$

Then expand (28) with the help of (18) to find

$$\tilde{M}(s) = \sum_{k=0}^{\infty} \sum_{\kappa} \left[-\frac{s}{\ln 2}\right]_\kappa \cdot \frac{E_{\boldsymbol{Q}}[\tilde{C}_\kappa(-\boldsymbol{Q})]}{k!}. \qquad (30)$$

Application of (29) leads to

$$\tilde{M}(s) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left[-\frac{s}{\ln 2}\right]_\kappa \cdot [\nu]_\kappa}{k!} \cdot \frac{\tilde{C}_\kappa(-\tilde{\Sigma}(\varepsilon)) \cdot \tilde{C}_\kappa(\Omega)}{\tilde{C}_\kappa(\boldsymbol{I})}. \qquad (31)$$

Now using definition (18) proves the theorem. *QED*

We emphasize again that Theorem 1 can be used for the analysis of systems with arbitrary antenna array sizes, arbitrary channel correlation and arbitrary noise and signal covariance matrices.

B. MGF in terms of scalar functions

The practical relevance of Theorem 1, where the MGF is essentially an infinite sum of zonal polynomials (there are no general formulas available for their calculation), can be established by the following theorem, which was given in [16] and independently in [17].

Theorem 2. Let **X** and **Y** be two diagonal matrices with $\boldsymbol{X} = \mathrm{diag}(x_1, \ldots, x_m)$ and $\boldsymbol{Y} = \mathrm{diag}(y_1, \ldots, y_m)$ with $x_1 > \ldots > x_m$ and $y_1 > \ldots > y_m$. Furthermore define

$$\Gamma_m(r) = \frac{\tilde{\Gamma}_m(r)}{\pi^{\frac{1}{2}m(m-1)}} = \prod_{i=1}^{m} \Gamma(r-i+1) \qquad (32)$$

and the Vandermonde determinant, which can be expressed as

$$\alpha_m(\boldsymbol{X}) = \prod_{i<j} (x_i - x_j). \qquad (33)$$

Furthermore, define the auxiliary function

$$\psi_q^{(m)}(b) = \prod_{i=1}^{m}\prod_{j=1}^{q}(b_j - i + 1)^{i-1}. \quad (34)$$

Then the hypergeometric functions of matrix argument can be expressed in terms of scalar hypergeometric functions

$$_p\tilde{\boldsymbol{F}}_q^{(m)}(a_1, \ldots, a_p; b_1, \ldots, b_q; \boldsymbol{X}, \boldsymbol{Y}) = \frac{(\Gamma_m(m) \cdot \psi_q^{(m)}(b))|\boldsymbol{D}|}{\alpha_m(\boldsymbol{X}) \cdot \alpha_m(\boldsymbol{Y}) \cdot \psi_p^{(m)}(a)} \quad (35)$$

with

$$\begin{gathered} d_{ij} = {}_pF_q(a_1 - m + 1, \ldots, a_p - m + 1; b_1 - m + 1, \ldots, b_q - m + 1; x_i y_j) \\ \boldsymbol{D} = [d_{ij}] \end{gathered}. \quad (36)$$

Using Theorem 2, we can directly derive the following corollary, which is important for the MIMO systems analyzed in this paper.

Corollary 1.

$$_2\tilde{\boldsymbol{F}}_0^{(m)}(a_1, a_2; \boldsymbol{X}, \boldsymbol{Y}) = \Gamma_m(m) \cdot \frac{|{}_2F_0(a_1 - m + 1, a_2 - m + 1;\ ; x_i y_j)|}{\alpha_m(\boldsymbol{X}) \cdot \alpha_m(\boldsymbol{Y}) \cdot \psi_2^{(m)}(a_1, a_2)}. \quad (37)$$

We emphasize that (37) is valid only for two m×m matrices **X** and **Y**, thus (37) can only be applied to (24) after artificially introducing a $\nu \times \nu$ MIMO system as outlined in (16) with a subsequent limiting process. We summarize the results in

Theorem 3. The MGF $M(s)$ of the MIMO channel mutual information according to (16) with parameters μ and ν is given by

$$M(s) = \lim_{\varepsilon \to 0} \Gamma_\nu(\nu) \cdot \frac{\left|{}_2F_0\left(-\frac{s}{\ln 2} - \nu + 1, 1;\ ; -\tilde{\sigma}_i(\varepsilon)\omega_j\right)\right|}{\alpha_\nu(-\tilde{\Sigma}(\varepsilon)) \cdot \alpha_\nu(\Omega) \cdot \psi_2^{(\nu)}\left(-\frac{s}{\ln 2}, \nu\right)}, \quad (38)$$

with the definition of $\tilde{\Sigma}(\varepsilon)$ in (15).

Via L'Hospital's rule it is possible to explicitly calculate the limit in (38) and we state without proof

Theorem 4. The MGF M(s) of the correlated MIMO channel mutual information according to (16) with parameters ν and μ is given by

$$M(s) = \frac{\chi}{\gamma^{\frac{\nu \cdot (\nu - 1)}{2}} \cdot |\Sigma|^{\nu - \mu}} \cdot \frac{|\Psi(s)|}{\alpha_\mu(\Sigma) \cdot \alpha_\nu(\Omega) \cdot \psi_2^{(\nu)}\left(-\frac{s}{\ln 2}, \nu\right)}, \quad (39)$$

with the auxiliary constant

$$\chi = \frac{\Gamma_\nu(\nu)}{(-1)^{\frac{\nu \cdot (\nu - 1)}{2}} \cdot (-1)^{\frac{(\nu - \mu) \cdot (\nu - \mu - 1)}{2}}} \quad (40)$$

and the $\nu \times \nu$ matrix

$$\Psi(s) = \begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \end{bmatrix}, \quad (41)$$

with the $\mu \times \nu$ matrix (i runs from 1 to μ and j from 1 to ν)

$$\Psi_1(s) = \left[{}_2F_0\left(-\frac{s}{\ln 2} - \nu + 1, 1;\ ; -\gamma\sigma_i\omega_j\right)\right], \quad (42)$$

the $(\nu - \mu) \times \nu$ matrix (i' runs from 1 to $\nu - \mu$ and j' from 1 to ν)

$$\Psi_2(s) = \left[(-\gamma\omega_{j'})^{i'-1}\left[-\frac{s}{\ln 2} - \nu + 1\right]_{i'-1}\right], \quad (43)$$

scalar hypergeometric function $_2F_0(a_1, a_2;\ ; z)$ [18], and Pochhammer's symbol according to (20).

V. Calculation of Ergodic Capacity

We focus on the ergodic capacity (first moment of mutual information) in this paper, however, arbitrary moments can be calculated from the MGF, allowing e.g. the approximation of the mutual information PDF by Gaussian or Gamma approximations. Note that from the MGF we derive the ergodic capacity C_{erg} by

$$C_{erg} = \left.\frac{d}{ds}M(s)\right|_{s=0}. \quad (44)$$

To this end, we can apply the following formula for the differentiation of a determinant

$$\frac{\partial}{\partial s}|\boldsymbol{X}(s)| = \sum_i |\boldsymbol{X}_i(s)|, \quad (45)$$

where $|\boldsymbol{X}_i(s)|$ is the determinant of matrix $\boldsymbol{X}$, where the ith column (or alternatively row) is differentiated with respect to s.

Theorem 5. The ergodic capacity of a correlated MIMO link with parameters ν and μ is given by

$$C_{erg}(\gamma) = \frac{\Gamma_\nu(\nu) \cdot (-1)^{\frac{(\nu - \mu) \cdot (\nu - \mu - 1)}{2}}}{\ln 2 \cdot \alpha_\mu(\Sigma) \cdot \alpha_\nu(\Omega) \cdot \gamma^{\frac{\nu \cdot (\nu - 1)}{2}} \cdot \prod_{k=1}^{\nu - 1} k^\nu} \cdot \sum_{l=1}^{\mu} \begin{vmatrix} \Xi(l) \\ \Psi_2(0) \end{vmatrix} \quad (46)$$

with the $\mu \times \nu$ matrix (i runs from 1 to μ and j from 1 to ν)

$$\Xi(l) = \begin{cases} \Gamma(\nu) \cdot \sigma_i^{\mu - 1} \cdot (\gamma\omega_j)^{\nu - 1} \cdot e^{\frac{1}{\gamma\sigma_i\omega_j}} \cdot E_1\left(\frac{1}{\gamma\sigma_i\omega_j}\right) & i = l \\ \sum_{k = \nu - \mu}^{\nu - 1} (-1)^k \cdot (\sigma_i)^{k - (\nu - \mu)} \cdot (\gamma\omega_j)^k \cdot [1 - \nu]_k & i \neq l \end{cases}, \quad (47)$$

where $E_1(x)$ is the exponential integral [18], and the definition of $\Psi_2(s)$ in (43).

Proof: See [21].

We note that the analysis can be generalized to cover the case of semi- or uncorrelated MIMO channels [22].

VI. Numerical Results

Without loss of generality, we study systems with white input signals of power E_s and additive white Gaussian noise of variance N_0 (other signal and noise covariances can easily be absorbed in an equivalent channel),

$$\boldsymbol{R}_{ss} = E_s \cdot \boldsymbol{I} \qquad \boldsymbol{R}_{nn} = N_0 \cdot \boldsymbol{I}. \quad (48)$$

Furthermore, in the following, we consider exponential correlation matrices at the receiver and the transmitter with

$$[\boldsymbol{R}_{\mathrm{RX/TX}}]_{ij} = (r_{\mathrm{RX/TX}})^{|i-j|}, \quad (49)$$

i.e. r_{RX} is the correlation coefficient between two neighboring receive antennas and r_{TX} models the correlation between two transmit antennas. We note that with the given channel model, the correlation between two antennas decreases exponentially with their distance. Finally, the SNR in dB is defined by

$$\gamma_{dB} \equiv 10 \cdot \log_{10} \frac{\rho \cdot E_s}{N_0} \qquad [dB], \qquad (50)$$

whereas the transmit power constraint is given by $\rho = T$.

Simulation results and theoretical curves according to Theorem 5 closely agree in Fig. 1 for a 4×4 system with correlated MIMO channel. Note that the correlation coefficient is the same in each case at the transmitter and receiver side $r_{RX}=r_{TX}=r$. As expected, the negative impact of channel correlation on ergodic capacity with uninformed transmitter can be observed.

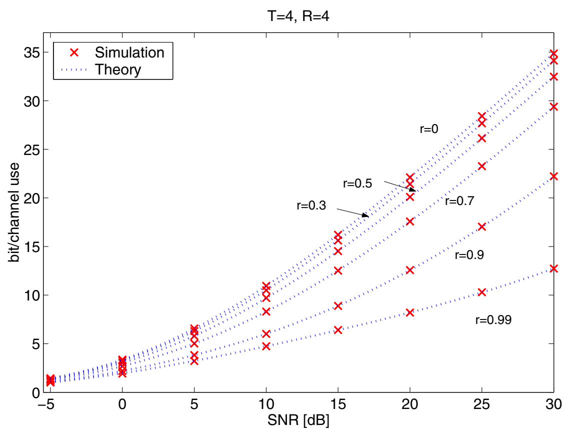

Fig. 1: Ergodic capacity, $r_{RX}=r_{TX}=r$, T=R=4

In Fig. 2 we have fixed the SNR and vary the correlation at the receiver side. The correlation coefficient at the transmitter is a parameter for the various curves. One can observe a significant loss in ergodic capacity for correlation coefficients greater than 0.7. Similar observations were made in the context of the performance analysis of smart antenna systems [19].

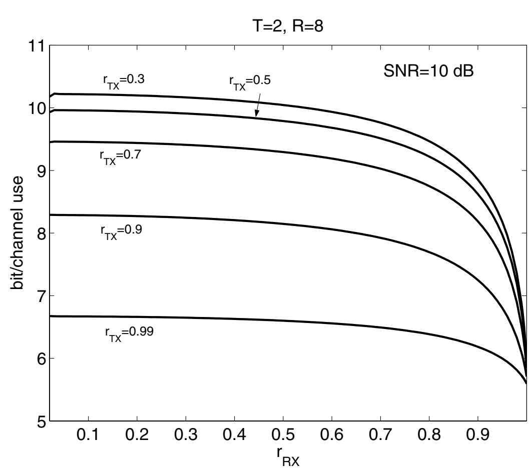

Fig. 2: Ergodic capacity, varying r_{RX}, fixed SNR

REFERENCES

[1] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Bell Labs Technical Memorandum*, June 1995

[2] G. J. Foschini, M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, no. 6, pp. 311-335, 1998

[3] D. Shiu, G. J. Foschini, M. J. Gans, J. M. Kahn, "Fading correlation and its effect on the capacity of multielement antenna systems," *IEEE Transactions on Communications*, vol. 48, no. 3, pp. 502-513, March 2000

[4] M. T. Ivrlac, T. P. Kurpjuhn, C. Brunner, W. Utschick, "Efficient use of fading correlations in MIMO systems," VTC, Oct. 2001

[5] S. Loyka, A. Kouki, "On the use of Jensen's inequality for MIMO channel capacity estimation," *Canadian Conference on Electrical and Computer Engineering*, vol. 1, pp. 475-480, 2001

[6] A. Grant, "Rayleigh fading multiple-antenna channels," *EURASIP Journal on Applied Signal Processing, Special Issue on Space-Time Coding (Part I)*, vol. 2002, no. 3, pp. 316-329, March 2002

[7] P. B. Rapajic, D. Popescu, "Information capacity of a random signature multiple-input multiple-output channel," *IEEE Transactions on Communications,* vol. 48, pp. 1245-48, Aug. 2000

[8] C.-N. Chuah, D. N. C. Tse, J. M. Kahn, R. A. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Transactions on Information Theory,* vol. 48, no.3, pp. 637-650, Mar. 2002

[9] Z. Wang, G. B. Giannakis, "Outage mutual information of space-time MIMO channels," *IEEE Transactions on Information Theory*, 2003, *submitted*

[10] Kang M., Alouni M.- S., "On the outage capacity of MIMO channels", *IEEE Transactions on Information Theory*, 2003, *submitted*

[11] Kermoal J. P., Schumacher L., Pedersen K. I., Mogensen P. E., Frederiksen F., "A stochastic MIMO radio channel model with experimental validation," *IEEE Journal on Selected Areas in Communications,* vol. 20, no. 6, pp. 1211-1226, Aug. 2002

[12] Gupta A. K., Nagar D. K., *Matrix variate distributions*, Chapman&Hall/CRC, 2000

[13] C. G. Khatri, "On certain distribution problems based on positive definite quadratic functions in normal vectors," *Ann. Math. Statist.,* vol. 37, pp. 468-479, 1966

[14] Lütkepohl H., *Handbook of matrices*, John Wiley&Sons, 1996

[15] James A. T., "Distributions of matrix variates and latent roots derived from normal samples," *Ann. Math. Statist.,* vol. 35, pp. 475-501, 1964

[16] Khatri C. G., "On the moments of traces of two matrices in three situations for complex multivariate normal populations," *Sankya: The Indian Journal of Statistics, Series A,* vol.32, part 1, pp. 65-80, 1970

[17] Gross K. I., Richards D., "Total positivity, spherical series, and hypergeometric functions of matrix argument," *Journal on Approximation Theory,* vol. 59, no.2, pp. 224-246, 1989

[18] M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions*, Dover Publications, Inc., New York 1964

[19] Salz J., Winters J. H., "Effect of fading correlation on adaptive arrays in digital mobile radio," *IEEE Transactions on Vehicular Technology*, vol. 43, no. 4, pp. 1049-1054, Nov. 1994

[20] Kiessling M., Viering I., Reinhardt M., Speidel J., "A closed form bound on correlated MIMO channel capacity," VTC, Sept. 2002

[21] Kiessling M., Speidel J., "Exact ergodic capacity of MIMO channels in correlated Rayleigh fading environments," *International Zurich Seminar on Communications,* Feb. 2004

[22] Kiessling M., Speidel J., Reinhardt M., "Unifying analysis of ergodic MIMO capacity in correlated Rayleigh fading environments," *European Wireless Conference, Feb.* 2004