Unifying Performance Analysis of Linear MIMO Receivers in Correlated Rayleigh Fading Environments

Mario Kiessling[1,2], Joachim Speidel[1]
[1]Institute of Telecommunications, University of Stuttgart, Germany
[2]Siemens AG, Information and Communication Mobile, Ulm, Germany
mario.kiessling@gmx.net, speidel@inue.uni-stuttgart.de

Abstract— In this paper, we present a novel approach for analyzing the SER of linear MIMO zero-forcing and minimum mean squared error receivers in correlated Rayleigh fading environments with fading correlation at transmit and - for the first time - receive antenna array. Specifically, we make use of certain complex Gaussian integrals for calculating the moment generating function of the subchannel SNR statistics. Using a well-known integral representation of the Gaussian Q function, it is then shown that the SER can be expressed in terms of a single integral. The novel approach is exact, non-asymptotic in the number of antenna elements, and unifies and complements existing partial solutions, thereby comprising the practically important case of multi-user smart antenna applications with optimum combining algorithm at the base station. Moreover, we present concise asymptotical expressions for the high SNR regime that allow for a simple assessment of the influence of fading correlation on the SER performance. Monte-Carlo simulations confirm the validity and accuracy of the analysis. The authors expect that the mathematical tools presented in this paper can be applied for solving a variety of other problems in communication theory.

I. Introduction

Research on the performance analysis of wireless MIMO systems in the majority of cases focuses on Shannon capacity (in particular ergodic capacity) and pairwise error probabilities (PEP) for maximum likelihood receivers. While ergodic capacity [1][2][22] and PEP [23] are well understood, only little is known about the symbol error rate (SER) performance of low-complexity linear MIMO receivers, especially in the presence of fading correlation at the receive antenna array. For uncorrelated Rayleigh fading, it was shown in [3] in the context of smart antenna systems that for zero-forcing (ZF) receivers, the subchannel signal to noise ratio (SNR) (for each user) follows a simple gamma distribution. This result was extended for MIMO systems to cover the case of fading correlation at the transmit antenna array in [4] and independently in [24]. On the other hand, many results are available on the analysis of minimum mean squared error (MMSE) processing (which is termed optimum combining in smart antenna literature) with spatially uncorrelated fading. The exact subchannel SINR distribution for users with different transmit powers was given in [5] based on a statistical result on certain quadratic forms in [6]. For equal-power interferers, an exact SER analysis was presented in [7], where the eigenvalue probability density function of complex Wishart matrices was used for the derivation [8].

However, to the author's best knowledge, no general exact analytical SER expressions can be found in literature for the case of spatial fading correlation at the receive antenna array. Available results for MMSE receivers are approximations [9] or are semi-analytic [10], thus still requiring lengthy Monte-Carlo simulations. For the special case of only two transmit and two receive antennas, exact SER formulas were given in [24] for ZF receivers and in [25] for MMSE receivers based on a random eigenvalue approach for systems with receive as well as transmit correlation. However, these results could not be generalized for an arbitrary number of transmit and receive antennas.

In this paper, for the first time we present fully analytic SER expressions for MIMO ZF and MMSE receivers for an arbitrary finite number of transmit and receive antennas with arbitrary fading correlation at the transmit as well as the receive antenna array. We emphasize that for the first time correlation at the receiver (a practically relevant case in multi-user beamforming scenarios) can be taken into account, which is not possible with other mathematical approaches. For the derivation, we present expressions for the subchannel SNR moment generating function (MGF) in terms of certain expected values of ratios of random determinants. As it appears that there are no results available in literature for calculating these expected values, we present closed form formulas that are derived by a novel mathematical approach. Specifically, we make use of certain complex Gaussian integrals [11][12] for the derivation. Based on the MGFs, we show that the SER can be given in terms of a single integral by using a well-known integral representation of the Gaussian Q function [13]. Furthermore, we present novel asymptotical SER expressions for the high SNR regime, which allow for a simple assessment of the influence of the various system parameters and especially fading correlation on the SER performance. In this regard, we point out the importance of the elementary symmetric functions of the eigenvalues of the receive correlation matrix.

Finally, Monte-Carlo simulations for different propagation environments show that the novel SER formulas exhibit a very high accuracy even for low SNR ranges of practical interest.

IEEE International Symposium on Spread Spectrum Techniques and Applications (ISSSTA)," Sydney, Australia, August 2004

II. SIGNAL AND CHANNEL MODEL

We consider a general flat fading MIMO link with transmit prefilter modeled by

$$y = HFs + n , \tag{1}$$

where **s** is the L×1 TX symbol vector, i.e. there are L independent data streams (subchannels) and **F** is a T×L linear matrix transmit prefilter. With the availability of adequate channel state information at the transmitter, a subchannel specific beamforming is possible via **F**, which can be used e.g. for minimizing the SER [26]. **H** is the R×T MIMO channel matrix with correlated Rayleigh fading elements, **n** is the R×1 noise vector, and **y** is the R×1 receive vector (see Fig. 1). By R≥L we denote the number of RX antennas and T is the number of TX antennas. Note that L can in general be smaller than the number of transmit antennas.

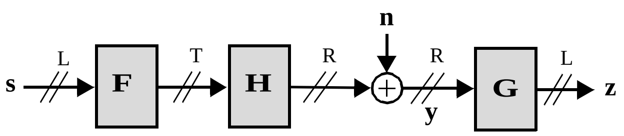

Fig. 1: System model

In the remainder of the paper, by **I** we denote an identity matrix, $\mathrm{diag}(x_1,\ldots x_N)$ is a diagonal matrix with elements $x_1,\ldots x_N$, $\mathrm{blkdiag}(\boldsymbol{D}_1, \ldots, \boldsymbol{D}_N)$ is a block diagonal matrix, eig(**X**) is a diagonal matrix of eigenvalues of **X**, vec(**X**) stacks the columns of matrix **X**, ⊗ is the Kronecker product, $\mathbf{X}^*$ means complex conjugate, $\mathbf{X}^T$ means transpose, $\mathbf{X}^H$ means Hermitian, $[\mathbf{X}]_{kk}$ is the kth diagonal element of **X**, x≅ means 'random variable (RV) x has the same distribution as', and $E_x[\ ]$ denotes expected value with respect to RV x.

We define the linear L×R zero-forcing receive matrix filter **G** (Fig. 1) and the L×1 vector **z** that is used for the subsequent symbol detection. In order to simplify notation, in the following we assume without loss of generality additive white Gaussian noise (AWGN), i.e. the noise covariance matrix is given by $\mathbf{R}_{nn}=N_0 \cdot \mathbf{I}$, where N_0 is the noise power. Furthermore, we assume that the symbol energy is normalized such that the covariance matrix of the symbol vector **s** is $\mathbf{R}_{ss}=E_s \cdot \mathbf{I}$, where E_s is the symbol energy (it is a simple exercise to take into account other choices of $\mathbf{R}_{nn}$ and $\mathbf{R}_{ss}$ by introducing an equivalent channel). The signal to noise ratio (SNR) is defined by

$$\gamma = E_s / N_0 . \tag{2}$$

Using a widely accepted channel model, the correlated MIMO channel can be described by the matrix product

$$H = A^H H_w B , \tag{3}$$

where $\mathbf{H}_w$ is a R×T matrix of complex i.i.d. Gaussian variables of unity variance and

$$AA^H = R_{RX} \qquad BB^H = R_{TX} , \tag{4}$$

where $\mathbf{R}_{RX}$ and $\mathbf{R}_{TX}$ is the long-term stable (normalized) receive and transmit correlation matrix, respectively. Furthermore, we introduce the eigenvalue decompositions (EVD)

$$R_{RX} = V_o \cdot O \cdot V_o^H . \tag{5}$$

with diagonal eigenvalue matrix $\boldsymbol{O} = \mathrm{diag}(o_1, \ldots, o_R)$ and

$$R_{TX} = V_{TX} \cdot \Lambda_{TX} \cdot V_{TX}^H . \tag{6}$$

We assume full rank correlation matrices in this paper, however, an extension is straightforward. Finally, it can be shown that the covariance of the channel model in (3) is given by

$$R_{HH} = E[\mathrm{vec}(H) \cdot (\mathrm{vec}(H))^H] = R_{TX}^* \otimes R_{RX} . \tag{7}$$

While the Kronecker product covariance structure in (7) is not the most general model for correlated MIMO channels, it is a good approximation of many practical propagation scenarios and its statistics are very well analyzed in multivariate statistical literature, thus motivating its use.

III. SUBCHANNEL SNR EXPRESSIONS

The following expressions for the subchannel SNR in terms of random quadratic forms in complex normal vectors are the basis for all derivations.

A. ZF receiver

Based on certain properties of the complex matrix variate normal distribution, we can derive the following

Lemma 1. The SNR statistics on subchannel k for a MIMO system with ZF receiver, L independent subchannels, R receive antennas, and correlated MIMO channel according to (3) with transmit as well as receive correlation can be expressed as a random quadratic form in complex normal vectors

$$\gamma_{SC,k} \cong \alpha_k \cdot u^H Q u , \tag{8}$$

where **u** is a R×1 column vector of unit variance i.i.d. complex Gaussian elements, the constants

$$\alpha_k = \frac{\gamma}{[C^{-1}]_{kk}} = \gamma \cdot \beta_k \qquad \beta_k = \frac{1}{[C^{-1}]_{kk}} , \tag{9}$$

the auxiliary matrix definition

$$C = F^H \cdot R_{TX} \cdot F , \tag{10}$$

and the R×R random matrix **Q**

$$Q = R_{RX} - R_{RX}\tilde{H}_w(\tilde{H}_w^H R_{RX} \tilde{H}_w)^{-1} \tilde{H}_w^H R_{RX} , \tag{11}$$

where $\tilde{\boldsymbol{H}}_w$ is a R×(L-1) matrix of complex Gaussian i.i.d. elements.

Proof: See [24].

B. MMSE receiver

Similar to the ZF case, we give an expression of the subchannel SNR in terms of a random quadratic form. While in principle an analysis of the general case is possible, due to the space limitation, in this paper we concentrate on long-term (LT) eigenmode (EM) transmission. This corresponds to a cellular scenario with multiple single-antenna unequal-power users and an antenna array at the base-station.

Lemma 2. With LT EM transmission, the subchannel SINR on subchannel k after MMSE processing at the receiver can be expressed as

$$\gamma_{SC,k} \cong c_k \cdot \boldsymbol{u}^H \boldsymbol{O}^{1/2}\left(\boldsymbol{O}^{1/2}\tilde{\boldsymbol{H}}_w \tilde{\boldsymbol{C}}_k \tilde{\boldsymbol{H}}_w^H \boldsymbol{O}^{1/2} + \frac{1}{\gamma}\boldsymbol{I}_R\right)^{-1} \boldsymbol{O}^{1/2}\boldsymbol{u} \quad (12)$$

where **u** is a R×1 column vector and $\tilde{\boldsymbol{H}}_w$ is a R×(L-1) matrix of complex Gaussian i.i.d. elements. The auxiliary L×L matrix defined in (10) simplifies for LT EM transmission with $\boldsymbol{F} = \tilde{\boldsymbol{V}}_{TX} \cdot \Phi$ ($\tilde{\boldsymbol{V}}_{TX}$ is a matrix of L eigenvectors according to the EVD in (6) and Φ is a diagonal power allocation (PA) matrix) to

$$\boldsymbol{C} = \boldsymbol{F}^H \cdot \boldsymbol{R}_{TX} \cdot \boldsymbol{F} = \Lambda_{TX} \cdot \Phi^2 = \mathrm{diag}(c_1, c_2, \ldots, c_L)\,. \quad (13)$$

We partition $\boldsymbol{C}$ with kth diagonal element c_k

$$\boldsymbol{C} = \mathrm{blkdiag}(\boldsymbol{C}_{1,k}, c_k, \boldsymbol{C}_{2,k}) \qquad \tilde{\boldsymbol{C}}_k = \mathrm{blkdiag}(\boldsymbol{C}_{1,k}, \boldsymbol{C}_{2,k})\,. \quad (14)$$

Proof: See [25].

IV. General MGF Based SER Calculation

In the following derivations of the ZF and MMSE receiver SER performance, we use the following conditional SER (conditioned on the subchannel SNR $\gamma_{SC,k}$) approximation for square M-QAM constellations (see e.g. [14])

$$P_{s,c}(\gamma_{SC,k}) \approx b \cdot \mathrm{erfc}(\sqrt{c\gamma_{SC,k}}) = 2b \cdot Q(\sqrt{2c\gamma_{SC,k}}) \quad (15)$$

with the modulation-dependent parameters

$$b = 2\left(1 - \frac{1}{\sqrt{M}}\right) \qquad c = \frac{3}{2(M-1)} \quad (16)$$

and constellation size M. Using an integral representation of the complementary error function together with the conditional SER approximation in (15) and (16), it is well known [13] that the average SER $P_{s,k}(\gamma)$ on subchannel k can be calculated via the subchannel SNR moment generating function (MGF)

$$M_{\gamma,k}(s) = E_{\gamma_{SC,k}}[\exp(-s\gamma_{SC,k})] \quad (17)$$

and the single integral

$$P_{s,k}(\gamma) = \frac{2b}{\pi} \cdot \int_0^{\frac{\pi}{2}} M_{\gamma,k}\left(\frac{c}{\sin^2\theta}\right) d\theta\,. \quad (18)$$

Furthermore, with Gray encoding and the assumption that a wrong symbol decision causes just one single bit error, which is fulfilled for higher SNR, we can approximate the bit error rate (BER) $P_{b,k}$ on subchannel k for a modulation size M by

$$P_{b,k}(\gamma) \approx \frac{1}{\log_2 M} \cdot P_{s,k}(\gamma)\,. \quad (19)$$

V. Subchannel SNR Moment Generating Functions

In this section we present expressions for the subchannel S(I)NR MGF for ZF and MMSE receivers in terms of expected values of ratios of certain random determinants. We then determine explicit closed-form expressions for the expected values. Furthermore, asymptotical formulas are given for the high SNR regime.

A. Zero-forcing receiver

Theorem 1. The MGF $M_{\gamma,k}(s)$ of the subchannel SNR in a correlated Rayleigh fading environment with MIMO ZF receiver is given by

$$M_{\gamma,k}(s) = \frac{1}{|\boldsymbol{I} + s\alpha_k \boldsymbol{O}|} \cdot E_{\tilde{\boldsymbol{H}}_w}\left[\frac{|\tilde{\boldsymbol{H}}_w^H \boldsymbol{O} \tilde{\boldsymbol{H}}_w|}{|\tilde{\boldsymbol{H}}_w^H \boldsymbol{O}(\boldsymbol{I} + s\alpha_k \boldsymbol{O})^{-1}\tilde{\boldsymbol{H}}_w|}\right], \quad (20)$$

where $\tilde{\boldsymbol{H}}_w$ is a R×(L-1) matrix of i.i.d. complex Gaussian distributed elements, α_k is defined in (9), and **O** given in (5).

Proof: Using the subchannel SNR expression in Lemma 1 we find

$$M_{\gamma,k}(s) = E_{\boldsymbol{u},\boldsymbol{Q}}[\exp(-s\alpha_k \cdot \boldsymbol{u}^H \boldsymbol{Q}\boldsymbol{u})]\,. \quad (21)$$

Explicitly calculating the expected value with respect to **u** using (49) and (51) results in

$$M_{\gamma,k}(s) = \frac{1}{|\boldsymbol{I} + s\alpha_k \boldsymbol{Q}|}\,. \quad (22)$$

Together with the expression of **Q** in (11) it can be shown that

$$M_{\gamma,k}(s) = \frac{1}{|\boldsymbol{I} + s\alpha_k(\boldsymbol{O} - \boldsymbol{O}\tilde{\boldsymbol{H}}_w(\tilde{\boldsymbol{H}}_w^H \boldsymbol{O}\tilde{\boldsymbol{H}}_w)^{-1}\tilde{\boldsymbol{H}}_w^H \boldsymbol{O})|}\,. \quad (23)$$

Now factoring out $|\boldsymbol{I} + c\alpha_k \boldsymbol{O}|$ in the denominator and multiplying nominator and denominator by $|\tilde{\boldsymbol{H}}_w^H \boldsymbol{O}\tilde{\boldsymbol{H}}_w|$ yields the result (20). *QED*.

It appears that there are no results available in literature on calculating the expected value of the ratio of determinants in (20) in the general case of arbitrary number of antenna elements and subchannels. In this paper, for the first time we determine exact closed form expressions by exploiting certain complex integrals. To this end, we first derive the following lemma.

Lemma 3. Let $\boldsymbol{G}$ be a m×n ($m \geq n$) matrix of i.i.d. complex Gaussian distributed elements, whereas

$$\tilde{\boldsymbol{A}} = \mathrm{diag}(a_1, a_2, \ldots, a_m) \qquad \tilde{\boldsymbol{B}} = \mathrm{diag}(b_1, b_2, \ldots, b_m) \quad (24)$$

are positive definite deterministic diagonal m×m matrices. Then

$$r(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, m, n) = E_{\boldsymbol{G}}\left[\frac{|\boldsymbol{G}^H\tilde{\boldsymbol{A}}\boldsymbol{G}|}{|\boldsymbol{G}^H\tilde{\boldsymbol{B}}\boldsymbol{G}|}\right] =$$

$$\sum_{\hat{\alpha}_n}\left(|\tilde{\boldsymbol{A}}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot \sum_{k=1}^{n}\int_0^\infty \frac{t^{n-1}}{(1+[\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_n}^{\hat{\alpha}_n}]_{kk}\cdot t)|\boldsymbol{I}+t\tilde{\boldsymbol{B}}|}dt\right), \quad (25)$$

where $|\tilde{\boldsymbol{A}}|_{\hat{\alpha}_n}^{\hat{\alpha}_n}$ is the determinant of the matrix that results from selecting the row and column subset $\hat{\alpha}_n$ from matrix $\tilde{\boldsymbol{A}}$, $\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_n}^{\hat{\alpha}_n}$ is the n×n matrix that results from selecting the row and column subset $\hat{\alpha}_n$ from matrix $\tilde{\boldsymbol{B}}$, and $[\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_n}^{\hat{\alpha}_n}]_{kk}$ is its kth diagonal element. In (25), the summation is over all possible $\binom{m}{n}$ subsets $\hat{\alpha}_n$. The integrals in (25) can be calculated in closed form and we obtain for the prototype integral with $b_j = [\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_n}^{\hat{\alpha}_n}]_{kk}$

$$I_{rat}(\boldsymbol{B}, n, j) = \int_0^\infty \frac{t^{n-1}}{(1+b_j\cdot t)|\boldsymbol{I}+t\boldsymbol{B}|}dt = K_1(\boldsymbol{B}, n, j) + K_2(\boldsymbol{B}, n, j), \quad (26)$$

where we have introduced

$$K_1(\boldsymbol{B}, n, j) = (-1)^n \cdot \sum_{\substack{k=1 \\ k \neq j}}^{m} \frac{\ln(b_k)\cdot b_k^{m-n}}{(b_k - b_i)\prod_{\substack{l=1 \\ l \neq k}}^{m}(b_k - b_l)} \quad (27)$$

and

$$K_2(\boldsymbol{B}, n, j) = \frac{b_j^{m-n-1}\cdot\left[1-\ln(b_j)\cdot\left(\sum_{\substack{l=1 \\ l\neq j}}^{m}\frac{b_l}{b_j-b_l}+n-1\right)\right]}{\prod_{\substack{l=1 \\ l\neq j}}^{m}(b_j-b_l)}. \quad (28)$$

For clarity, the index j is given by $j = \hat{\alpha}_n(k)$, which is the kth element of subset $\hat{\alpha}_n$.

Proof: See Appendix B for an outline.

With the help of Lemma 3 we can directly get from Theorem 1 the following closed form expression for the subchannel SNR MGF.

Theorem 2. The MGF $M_{\gamma,k}(s)$ of the subchannel SNR in a correlated Rayleigh fading environment with MIMO ZF receiver and AWGN is explicitly given by

$$M_{\gamma,k}(s) = \frac{1}{|\boldsymbol{I}+s\gamma\beta_k\boldsymbol{O}|}\cdot r(\boldsymbol{O}, \boldsymbol{O}(\boldsymbol{I}+s\gamma\beta_k\boldsymbol{O})^{-1}, R, L-1), \quad (29)$$

where $r(\boldsymbol{A}, \boldsymbol{B}, m, n)$ is given in Lemma 3, β_k is defined in (9), and the R×R matrix $\boldsymbol{O} = \mathrm{eig}(\boldsymbol{R}_{RX})$.

We note that the exact MGF in (29) can be approximated very well by a tight bound. We refer the reader to [27] for more information. In the high SNR regime (29) considerably simplifies and we get

Theorem 3. In the high SNR regime, the MGF reads

$$\bar{M}_{\gamma,k}(s) = \frac{\mathrm{tr}_{L-1}(\boldsymbol{O})}{|\boldsymbol{O}|\cdot\binom{R}{L-1}}\cdot(s\gamma\beta_k)^{-N} \quad (30)$$

with diversity parameter $N = R - L + 1$, β_k defined in (9), and nth order elementary symmetric function of the eigenvalues of matrix $\boldsymbol{X}$ given by

$$\mathrm{tr}_n(\boldsymbol{X}) = \sum_{\hat{\alpha}_n}|\mathrm{eig}(\boldsymbol{X})|_{\hat{\alpha}_n}^{\hat{\alpha}_n}. \quad (31)$$

Proof: The solution was given in [27] using the so-called Laplace approximation. We note that a derivation based on complex integrals (see appendix) is also possible.

B. MMSE receiver

Using the same proceeding as in case of the ZF receiver, we find (without proof)

Theorem 4. With long-term eigenmode transmission and AWGN, the MGF of the subchannel SINR after MMSE receive processing for subchannel k is given by

$$M_\gamma(s,k) = \frac{1}{|\boldsymbol{I}+sc_k\gamma\boldsymbol{O}|}\cdot E_{\tilde{\boldsymbol{H}}_w}\left[\frac{|\gamma\tilde{\boldsymbol{H}}_w^H\boldsymbol{O}\tilde{\boldsymbol{H}}_w\tilde{\boldsymbol{C}}_k+\boldsymbol{I}_{L-1}|}{\left|\tilde{\boldsymbol{H}}_w^H\left(\frac{1}{\gamma}\boldsymbol{O}^{-1}+sc_k\cdot\boldsymbol{I}\right)^{-1}\tilde{\boldsymbol{H}}_w\tilde{\boldsymbol{C}}_k+\boldsymbol{I}_{L-1}\right|}\right], \quad (32)$$

where $\tilde{\boldsymbol{H}}_w$ is a R×(L-1) matrix of i.i.d. complex Gaussian distributed entries, with $\boldsymbol{O} = \mathrm{eig}(\boldsymbol{R}_{RX})$ from (5), and the definitions of Lemma 2.

In the analysis of the MMSE receiver, we have to differentiate between various propagation scenarios. Due to the space limitation, in this paper we therefore exemplarily focus on the case of receive correlation only. A complete derivation will be presented in the journal version of this paper. Again we need a lemma for calculating the expected value in (32). We state without proof (the derivation is again based on the complex integrals in the appendix)

Lemma 4. Let $\boldsymbol{G}$ be a m×n ($m \geq n$) matrix of i.i.d. complex Gaussian distributed elements, whereas

$$\tilde{\boldsymbol{A}} = \mathrm{diag}(a_1, a_2, \ldots, a_m) \qquad \tilde{\boldsymbol{B}} = \mathrm{diag}(b_1, b_2, \ldots, b_m) \quad (33)$$

are positive definite deterministic diagonal m×m matrices. Then

$$\bar{r}_1(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, m, n) = E_{\boldsymbol{G}}\left[\frac{\left|\boldsymbol{I}_n + \boldsymbol{G}^H\tilde{\boldsymbol{A}}\boldsymbol{G}\right|}{\left|\boldsymbol{I}_n + \boldsymbol{G}^H\tilde{\boldsymbol{B}}\boldsymbol{G}\right|}\right] = \sum_{k=0}^{n}\binom{n}{k}\sum_{\hat{\alpha}_k}|\tilde{\boldsymbol{A}}|_{\hat{\alpha}_k}^{\hat{\alpha}_k} \cdot \bar{r}_{k,\alpha}, \tag{34}$$

with

$$\bar{r}_{\alpha,k} = \frac{\Gamma(k+1)}{\Gamma(n+1)} \cdot \left[(n-k)\int_0^\infty \frac{t^{n-1}e^{-t}}{|\boldsymbol{I} + t\tilde{\boldsymbol{B}}|}dt + \left(\sum_{l=1}^{k}\int_0^\infty \frac{t^{n-1}e^{-t}}{(1 + t \cdot [\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_k}^{\hat{\alpha}_k}]_{kk})|\boldsymbol{I} + t\tilde{\boldsymbol{B}}|}dt\right)\right]. \tag{35}$$

where $|\tilde{\boldsymbol{A}}|_{\hat{\alpha}_k}^{\hat{\alpha}_k}$ is the determinant of the matrix that results from selecting the row and column subset $\hat{\alpha}_k$ from matrix $\tilde{\boldsymbol{A}}$, $\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_k}^{\hat{\alpha}_k}$ is the k×k matrix that results from selecting the row and column subset $\hat{\alpha}_k$ from matrix $\tilde{\boldsymbol{B}}$, and $[\{\tilde{\boldsymbol{B}}\}_{\hat{\alpha}_k}^{\hat{\alpha}_k}]_{ll}$ is its lth diagonal element.

We note that the integral in (35) can be calculated in terms of the exponential integral by proper decomposition in partial fractions, however, the calculations are lengthy and thus beyond the scope of this paper. With Lemma 4 we get

Theorem 5. The MGF $M_{\gamma,k}(s)$ of the subchannel SINR in a correlated Rayleigh fading environment with receive correlation only, $\boldsymbol{C} = \boldsymbol{I}$, AWGN, and MIMO MMSE receiver is explicitly given by

$$M_{\gamma,k}(s) = \frac{1}{|\boldsymbol{I} + s\gamma\boldsymbol{O}|} \cdot \bar{r}_1\left(\gamma\boldsymbol{O}, \left(\frac{1}{\gamma}\boldsymbol{O}^{-1} + s \cdot \boldsymbol{I}\right)^{-1}, R, L-1\right), \tag{36}$$

where $\bar{r}_1(\boldsymbol{A}, \boldsymbol{B}, m, n)$ is given in Lemma 4 and the R×R matrix $\boldsymbol{O} = \operatorname{eig}(\boldsymbol{R}_{RX})$.

Again we present results for the high SNR regime.

Theorem 6. With eigenmode transmission, the MGF $M_\gamma(s,k)$ of the SINR on subchannel k after MMSE receive processing given in Theorem 4 can be approximated in the high SNR region by

$$M_\gamma(s,k) \approx \bar{M}_\gamma(s,k) = \frac{|\tilde{\boldsymbol{C}}_k| \cdot \gamma^{-N}}{|sc_k\boldsymbol{O}|} \cdot E_{\tilde{\boldsymbol{H}}_w}\left[\frac{|\tilde{\boldsymbol{H}}_w^H\boldsymbol{O}\tilde{\boldsymbol{H}}_w|}{\left|\frac{1}{sc_k}\tilde{\boldsymbol{H}}_w^H\tilde{\boldsymbol{H}}_w\tilde{\boldsymbol{C}}_k + \boldsymbol{I}_{L-1}\right|}\right] \tag{37}$$

with the definitions of Lemma 2, diversity parameter $N = R - L + 1$, and $\boldsymbol{O} = \operatorname{eig}(\boldsymbol{R}_{RX})$.

For calculating the expected value in (37) we need (without proof)

Lemma 5. Let **G** be i.i.d. m×n matrix ($m \geq n$) of complex Gaussian elements. Then

$$\bar{r}_2(a, \boldsymbol{M}) = E_{\boldsymbol{G}}\left[\frac{|\boldsymbol{G}^H\boldsymbol{M}\boldsymbol{G}|}{|\boldsymbol{I} + a \cdot \boldsymbol{G}^H\boldsymbol{G}|}\right] = \operatorname{tr}_n(\boldsymbol{M}) \cdot n \cdot \int_0^\infty \frac{t^{n-1} \cdot e^{-t}}{(1+at)^{m+1}}dt \tag{38}$$

for m×m deterministic diagonal matrix **M** and constant a. Again, $\operatorname{tr}_n(\boldsymbol{M})$ denotes the nth elementary symmetric function.

We note that the integral in (38) can be written in terms of incomplete Gamma functions by applying integration by parts

$$\int_0^\infty \frac{t^{n-1} \cdot e^{-t}}{(1+at)^{m+1}}dt = \sum_{k=0}^{n-1}(-1)^k \cdot \binom{n-1}{k} \cdot \frac{e^{1/a}}{a^{m+k+1}} \cdot \Gamma\left(n-m-k-1, \frac{1}{a}\right). \tag{39}$$

Using (38) in (37), we can directly find

Theorem 7. The MGF $M_\gamma(s,k)$ of the SINR on subchannel k after MMSE receive processing can be approximated in the high SNR region for a channel with $\boldsymbol{C} = \boldsymbol{I}$ by

$$\bar{M}_\gamma(s,k) = \frac{(L-1) \cdot \operatorname{tr}_{L-1}(\boldsymbol{O})}{|\boldsymbol{O}| \cdot s^R} \cdot \left[\int_0^\infty \frac{t^{L-2} \cdot e^{-t}}{\left(1 + \frac{1}{s}t\right)^{R+1}}dt\right] \cdot \gamma^{-N}. \tag{40}$$

VI. Linear Receiver SER Calculation

With the exact closed form MGF expressions in (29), (36), and formula (18), the SER calculation can be reduced to a single integral with finite integration limits that can be evaluated in a straightforward manner with standard math software like e.g. Matlab. Results will be presented below in the numerical results section at the end of this paper. We note that further analytical calculations of the SER are possible, however, the resulting expressions are lengthy and thus beyond the scope of this paper. Therefore, we restrict the explicit analysis to the high SNR regime. The following theorems follow from direct integration of (30)(37) with (18) and are stated without proof.

A. ZF receiver

Theorem 8. The SER asymptotics on subchannel k $\bar{P}_{s,k}(\gamma)$ in the high SNR region with fully correlated (transmit as well as receive correlation) Rayleigh fading MIMO channel with AWGN and ZF receiver is given by

$$\bar{P}_{s,k}(\gamma) = b \cdot \frac{\operatorname{tr}_{L-1}(\boldsymbol{O})}{|\boldsymbol{O}| \cdot \binom{R}{L-1}} \cdot \frac{1 \cdot 3 \cdot 5 \ldots \cdot (2N-1)}{2^N \cdot \Gamma(N+1)} \cdot (c\beta_k)^{-N} \cdot \gamma^{-N}, \tag{41}$$

where b and c are modulation-dependent parameters according to (16), $N = R - L + 1$ is the diversity of the system, β_k is defined in (9), and $\boldsymbol{O} = \operatorname{eig}(\boldsymbol{R}_{RX})$. Again, $\operatorname{tr}_n(\boldsymbol{X})$ is the elementary symmetric function of order n of the eigenvalues of matrix **X**.

The diversity of N of the system becomes obvious in (41), where it is the negative exponent of SNR. Moreover, the influence of transmit (implicitly given in β_k) and receive correlation matrix is independent in (41), which is a consequence of the Kronecker correlation matrix channel model.

B. MMSE receiver

Theorem 9. The SER $\bar{P}_{s,k}$ on subchannel k after MMSE receive processing can be approximated in the high SNR region for a channel with uncorrelated fading at the transmitter ($\boldsymbol{C} = \boldsymbol{I}$) by

$$\bar{P}_{s,k} = \frac{b}{c^N} \cdot \frac{\Gamma(L) \cdot \operatorname{tr}_{L-1}(\boldsymbol{O})}{|\boldsymbol{O}| \cdot 4^R} \cdot \binom{2R}{R} \cdot U\left(L-1, -N+\frac{1}{2}, c\right) \cdot \gamma^{-N}, \tag{42}$$

where $U(a, b, z)$ is Kummer's U function.

If we compare (42) and (41), it can be realized that as expected the SER asymptotics of the MMSE receiver are just a shifted version (by a constant factor) of the ZF asymptotics.

VII. Numerical Results

Without loss of generality, we study a system with additive white Gaussian noise of variance N_0, i.e. $\boldsymbol{R}_{nn} = N_0 \cdot \boldsymbol{I}$. In the following, we consider exponential correlation matrices at the receiver and the transmitter with

$$[\boldsymbol{R}_{RX/TX}]_{ij} = (r_{RX/TX})^{|i-j|}, \tag{43}$$

i.e. r_{RX} is the correlation coefficient between two neigboring receive antennas and r_{TX} models the correlation between two transmit antennas. Obviously, the correlation between two antennas decreases exponentially with their distance. Moreover, the SNR in the following figures is defined by

$$SNR = 10 \cdot \log_{10}\left(\frac{T \cdot E_b}{N_0}\right) \qquad [dB], \tag{44}$$

where E_b is the energy per information bit.

Simulation results and theoretical curves closely agree in Fig. 2 for a system with T=4 transmit, R=6 receive antennas, and ZF receiver. The correlation at the receiver is kept constant r_{RX}=0.7, while the correlation at transmitter increases from left to right.

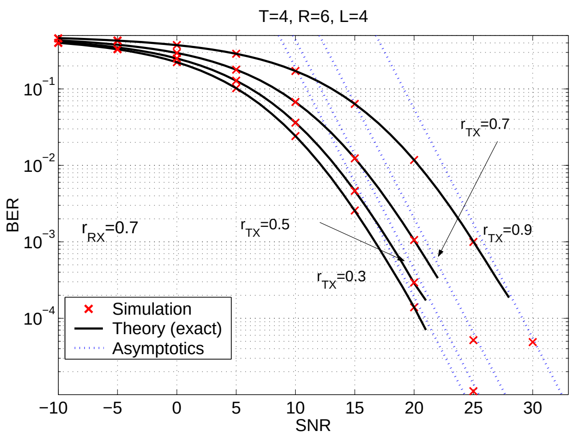

Fig. 2: ZF BER, T=L=4, R=6, r_{TX}={0.3,0.7,0.9}, r_{RX}=0.7

In Fig. 2 we have depicted Monte-Carlo simulation results, asymptotics according to Theorem 8, and exact analytical curves according to (18) in combination with Theorem 2.

Similar results are shown in Fig. 3 for a MMSE receiver and a semi-correlated channel with correlation at the receiver only. The asymptotics result from Theorem 9 and the exact analytical curves from (18) in combination with Theorem 5.

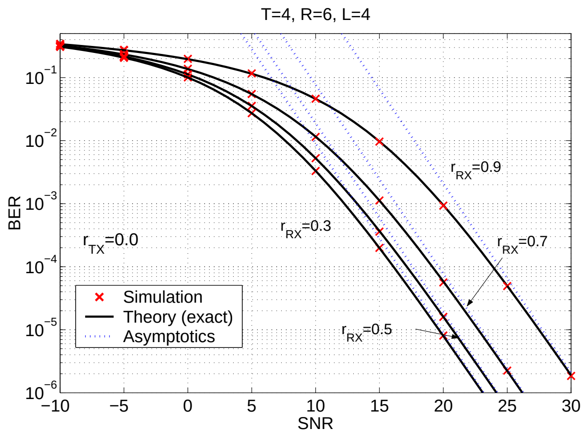

Fig. 3: MMSE BER, T=L=4, R=6, r_{RX}={0.3,0.5,0.7,0.9}, r_{TX}=0.0

Conclusion

We have presented a novel mathematical approach based on certain complex integrals to the SER performance analysis of linear MIMO receivers. Based on the new approach, we have for the first time presented exact subchannel SNR MGFs for ZF and MMSE receivers in correlated Rayleigh fading environments. Numerical results have demonstrated a tight match between theoretical SER analysis based on the closed-form MGFs and Monte-Carlo simulation results. Moreover, we have derived low-complexity high SNR asymptotics, which exhibit a direct dependency on certain elementary symmetric functions of the eigenvalues of the channel correlation matrices.

Appendix A

A. Complex integrals - definitions

In this paper, we use the following notation for the complex matrix integral measure for complex M×N matrix $\boldsymbol{X} = [x_{mn}]$

$$D_c\boldsymbol{X} = \prod_{m=1}^{M}\prod_{n=1}^{N}\frac{dRe\{x_{mn}\} \cdot dIm\{x_{mn}\}}{\pi}. \tag{45}$$

Furthermore, we simply write for a function $f(\boldsymbol{X})$

$$\int f(\boldsymbol{X}) D_c\boldsymbol{X} = \left\{ \begin{array}{l} \int\limits_{Re\{x_{11}\}=-\infty}^{Re\{x_{11}\}=\infty} \cdots \int\limits_{Re\{x_{mn}\}=-\infty}^{Re\{x_{mn}\}=\infty} \int\limits_{Im\{x_{11}\}=-\infty}^{Im\{x_{11}\}=\infty} \cdots \int\limits_{Im\{x_{mn}\}=-\infty}^{Im\{x_{mn}\}=\infty} f([x_{mn}]) \\ \cdots \prod_{m=1}^{M}\prod_{n=1}^{N}\frac{dRe\{x_{mn}\} \cdot dIm\{x_{mn}\}}{\pi} \end{array} \right. . \tag{46}$$

B. Some Gaussian integrals

The following results are given e.g. in [11][12][15][16]. For complex column m×1 vectors **x**,**a**,**b** and complex m×m matrix **A** it can be shown that

$$\int e^{-(\mathbf{x}^H\boldsymbol{A}\mathbf{x} + \boldsymbol{a}^H\mathbf{x} + \mathbf{x}^H\boldsymbol{b})} D_c\mathbf{x} = \frac{1}{|\boldsymbol{A}|} \cdot e^{\boldsymbol{a}^H\boldsymbol{A}\boldsymbol{b}}. \tag{47}$$

The result in (47) can be generalized and for complex m×n matrices **X**,**A**,**B**, m×m matrix **M**, and n×n matrix **N** ($n \le m$) we get

$$\int e^{-\mathrm{tr}(\boldsymbol{N}\boldsymbol{X}^H\boldsymbol{M}\boldsymbol{X} + \boldsymbol{A}^H\boldsymbol{X} + \boldsymbol{X}^H\boldsymbol{B})} D_c\boldsymbol{X} = \frac{1}{|\boldsymbol{N} \otimes \boldsymbol{M}|} \cdot e^{\mathrm{tr}(\boldsymbol{N}^{-1}\boldsymbol{A}^H\boldsymbol{M}^{-1}\boldsymbol{B})} . \quad (48)$$

An important special case of (48) is

$$\int e^{-\boldsymbol{x}^H\boldsymbol{M}\boldsymbol{x}} D_c\boldsymbol{x} = \frac{1}{|\boldsymbol{M}|} . \quad (49)$$

One can establish the following integral identity with scalar function $f(\boldsymbol{x}^H\boldsymbol{x})$ and m×1 complex vector $\boldsymbol{x}$

$$\int f(\boldsymbol{x}^H\boldsymbol{x}) \cdot e^{-\boldsymbol{x}^H\boldsymbol{x}} D_c\boldsymbol{x} = \frac{1}{\Gamma(m)} \cdot \int_0^\infty f(t) \cdot t^{m-1} \cdot e^{-t} dt . \quad (50)$$

The identity follows from the fact that (50) is the expected value of $f(\boldsymbol{x}^H\boldsymbol{x})$ with respect to an i.i.d. complex Gaussian distributed m×1 vector **x**, i.e. $t = \boldsymbol{x}^H\boldsymbol{x}$ has a Gamma distribution with m degrees of freedom.

C. Complex Gaussian distribution and associated integrals

Let **G** be an i.i.d. m×n matrix ($m \ge n$) of zero mean complex Gaussian distributed elements. Then the joint probability density function (PDF) is (including the integration measure)

$$p(\boldsymbol{G}) = e^{-\mathrm{tr}(\boldsymbol{G}^H\boldsymbol{G})} D_c\boldsymbol{G} . \quad (51)$$

Without going into the details, by using results of [17] it can be shown that

$$\int |\boldsymbol{G}^H\boldsymbol{G}| \cdot e^{-\mathrm{tr}(\Sigma^{-1}\boldsymbol{G}^H\boldsymbol{G})} D_c\boldsymbol{G} = \binom{m}{n} \cdot n! \cdot |\Sigma|^{n+1} . \quad (52)$$

Based on certain expected values in [18][19], we can derive for rank 1 matrix $\boldsymbol{M}$

$$\int |\boldsymbol{G}^H\boldsymbol{G}| \cdot e^{-\mathrm{tr}((\boldsymbol{G}-\boldsymbol{M})^H(\boldsymbol{G}-\boldsymbol{M}))} D_c\boldsymbol{G} = \frac{1}{\Gamma(m-n+1)}[\Gamma(m+1) + \Gamma(m) \cdot \mathrm{tr}(\boldsymbol{M}\boldsymbol{M}^H)] . \quad (53)$$

APPENDIX B

We use (49) and get with n×1 complex vector **x** for the expected value of the ratio of random determinant in (25)

$$r = E_{\boldsymbol{G}}\left[|\boldsymbol{G}^H\tilde{\boldsymbol{A}}\boldsymbol{G}| \cdot \int e^{-\boldsymbol{x}^H(\boldsymbol{G}^H\tilde{\boldsymbol{B}}\boldsymbol{G})\boldsymbol{x}} D_c\boldsymbol{x}\right] . \quad (54)$$

Using (48) for separating $\boldsymbol{G}$ and $\boldsymbol{G}^H$ (this is necessary for integrating out $\boldsymbol{G}$ later), we can express in (54)

$$e^{-\boldsymbol{x}^H\boldsymbol{G}^H\tilde{\boldsymbol{B}}\boldsymbol{G}\boldsymbol{x}} = \int (e^{-\boldsymbol{y}^H\boldsymbol{y}} e^{-\mathrm{tr}(i\boldsymbol{x}\boldsymbol{y}^H\tilde{\boldsymbol{B}}^{1/2}\boldsymbol{G} + \boldsymbol{G}^H\tilde{\boldsymbol{B}}^{1/2}\boldsymbol{y}\boldsymbol{x}^H i)}) D_c\boldsymbol{y} . \quad (55)$$

with m×1 complex vector **y**. The determinant in (54) can be split into a sum of determinants [20][21]

$$|\boldsymbol{G}^H\tilde{\boldsymbol{A}}\boldsymbol{G}| = \sum_{\hat{\alpha}_n} |\tilde{\boldsymbol{A}}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot |\boldsymbol{G}^H|_{\hat{\alpha}_n}^{\{1,\ldots,n\}} \cdot |\boldsymbol{G}|_{\{1,\ldots,n\}}^{\hat{\alpha}_n} , \quad (56)$$

where $|\boldsymbol{X}|_{\hat{\beta}_n}^{\hat{\alpha}_n}$ denotes the determinant of the matrix resulting from selecting row subset $\hat{\alpha}_n$ and column subset $\hat{\beta}_n$ from matrix $\boldsymbol{X}$. After completing the square in the exponential of (55) and proper partitioning of $\boldsymbol{G}$ we can use integral identities (47) and (53) for integrating out $\boldsymbol{G}$. Then integrating with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$, where we can use (50), we arrive after simplifying at (25).

By proper contour integration and the residue theorem, it can be shown that the integral in (26) is given by

$$I_{rat}(\tilde{\boldsymbol{B}}, n, j) = -\sum_l \mathrm{Res}_l\left(\frac{\ln(|z|) \cdot z^{n-1}}{(1+zb_j)|\boldsymbol{I} + z\tilde{\boldsymbol{B}}|}\right)\Bigg|_{z = z_l} , \quad (57)$$

where the sum is over all residues at the poles z_l of the indicated function. To this end, note that there are single poles at $z_k = -1/b_k$ for all $k = \{1, \ldots, m\}\backslash j$ and a double pole at $z_j = -1/b_j$. At the single poles we get

$$\mathrm{Res}_k\left(\frac{\ln(|z|) \cdot z^{n-1}}{(1+zb_i)|\boldsymbol{I} + z\tilde{\boldsymbol{B}}|}\right)\Bigg|_{z = -\frac{1}{b_k}} = \frac{\ln(|z|) \cdot z^{n-1} \cdot \left(z - \left(-\frac{1}{b_k}\right)\right)}{(1+zb_i)|\boldsymbol{I} + z\tilde{\boldsymbol{B}}|}\Bigg|_{z = -\frac{1}{b_k}} . \quad (58)$$

After some straightforward manipulations we find (27). On the other hand, at the double pole $z_j = -1/b_j$ we find

$$\mathrm{Res}_j\left(\frac{\ln(|z|) \cdot z^{n-1}}{(1+zb_j)|\boldsymbol{I} + z\tilde{\boldsymbol{B}}|}\right)\Bigg|_{z = -\frac{1}{b_j}} = \frac{d}{dz}\left(\frac{\ln(|z|) \cdot z^{n-1} \cdot \left(z - \left(-\frac{1}{b_j}\right)\right)^2}{(1+zb_j)|\boldsymbol{I} + z\tilde{\boldsymbol{B}}|}\right)\Bigg|_{z = -\frac{1}{b_j}} . \quad (59)$$

Now carrying out the differentiation and simplifying the results, we can establish (28). *QED*.

REFERENCES

[1] Telatar I. E., “Capacity of multi-antenna Gaussian channels,” *Bell Labs Technical Memorandum*, June 1995

[2] Kang M., Alouini M.-S., “On the outage capacity of MIMO channels,” *IEEE Transactions on Information Theory*, 2003, *submitted*

[3] Winters J. H., Salz J., Gitlin R. D., ”The impact of antenna diversity on the capacity of wireless communication systems,” *Transactions on Communications*, vol. 42, no. 2/3/4, pp. 1740-1751, Feb./March/April 1994

[4] Gore D., Heath R. W. Jr., Paulraj A., "Transmit selection in spatial multiplexing systems," *IEEE Communications Letters*, vol. 6, no. 11, Nov. 2002

[5] H. Gao, P. J. Smith, M.V. Clark, "Theoretical reliability of MMSE linear diversity combining in Rayleigh-fading additive interference channels," *IEEE Transactions on Communications*, vol. 46, no. 5, pp. 666-672, May 1998

[6] C. G. Khatri, “On certain distribution problems based on positive definite quadratic functions in normal vectors,” *Ann. Math. Statist.*, vol. 37, pp. 468-479, 1966

[7] Mallik R. K., Win M. Z., Chiani M., “Exact analysis of optimum combining in interference and noise over a Rayleigh fading channel,” ICC, May 2002

[8] James A. T., “Distributions of matrix variates and latent roots derived from normal samples,” *Ann. Math. Statist.*, vol. 35, pp. 475-501, 1964

[9] Cui J., Sheikh A. U. H., Falconer D. D., “BER analysis of optimum combining and maximal ratio combining with channel correlation for dual antenna systems,” VTC, pp. 150-154, May 1997

[10] Pham T. D., Balmain K. G., “Multipath performance of adaptive antennas with multiple interferers and correlated fadings,” *IEEE Transactions on Vehicular Technology*, vol. 48, no. 2, March 1999

[11] Dogandzic A., "Chernoff bounds on the pairwise error probabilities of space-time codes," *IEEE Transactions on Information Theory*, vol 49, no. 5, May 2003

[12] Moustakas A. L., Simon S. H., Sengupta A. M., "MIMO capacity through correlated channels in the presence of correlated interferers and noise: a (not so) large N analysis," *IEEE Transactions on Information Theory*, vol. 49, no. 10, Oct. 2003

[13] Simon M. K., Alouini M.-S., *Digital Communication over generalized fading channels: a unified approach to the performance analysis*, Wiley & Sons, Inc., 2000

[14] Proakis J., *Digital Communications*, McGraw Hill, 1995

[15] Harville D. A., *Matrix algebra from a statistician’s perspective*, Springer, 1997

[16] Graybill F. A., *Matrices with applications in statistics*, Wadsworth, 1983

[17] Goodman N. R., "The distribution of the determinant of a complex Wishart distributed matrix," *Annals of Mathematical Statistics*, vol. 34, pp. 178-180, 1963

[18] Shah B. K., Khatri C. G., “Proof of conjectures about the expected values of the elementary symmetric functions of a noncentral Wishart matrix”, *The Annals of Statistics*, vol. 2, no. 4, pp. 833-836, July 1974

[19] de Waal D. J., "On the expected values of the elementary symmetric functions of a noncentral Wishart matrix," *Annals of Mathematical Statistics*, vol. 43, pp. 344-347, 1972

[20] Aitken A. C., *Determinants and matrices*, Oliver and Boyd, 1964

[21] Browne E. T., *Introduction to the theory of determinants and matrices*, University of North Carolina Press, 1958

[22] Kiessling M., Speidel J., Reinhardt M., “Unifying analysis of ergodic MIMO capacity in correlated Rayleigh fading environments,” *European Wireless Conference*, Feb. 2004

[23] Kiessling M., Speidel J., Geng N., Reinhardt M., “Performance analysis of MIMO maximum likelihood receivers with channel correlation, colored Gaussian noise, and linear prefiltering,“ ICC, 2003, May 2003

[24] Kiessling M., Speidel J., “Analytical performance of MIMO zero-forcing receivers in correlated Rayleigh fading environments”, SPAWC, June 2003

[25] Kiessling M., Speidel J., "Analytical performance of MIMO MMSE receivers in correlated Rayleigh fading environments", VTC, Oct. 2003

[26] Kiessling M., Speidel J., “Statistical prefilter design for MIMO ZF and MMSE receivers based on majorization theory,” ICASSP, May 2004

[27] Kiessling M., Speidel J., "Asymptotically tight bound on SER of MIMO zero-forcing receivers with correlated fading," *IEEE International Symposium on Information Theory*, ISIT, June 2004, *submitted*