Asymptotics of Ergodic MIMO Capacity in Correlated Rayleigh Fading Environments

Mario Kiessling[1,2], Joachim Speidel[1], Alexander Boronka[1]
[1]Institute of Telecommunications, University of Stuttgart, Germany
[2]Siemens AG, Information and Communication Mobile, Ulm, Germany
mario.kiessling@gmx.net, {speidel,boronka}@inue.uni-stuttgart.de

Abstract—Using recent analytical results on the exact ergodic capacity of correlated MIMO channels with transmit as well as receive correlation in flat Rayleigh fading environments, we derive simple asymptotic expressions for ergodic capacity in the low and high SNR region. The asymptotics allow for a simple assessment of the influence of the various system parameters and fading correlation on ergodic capacity. Furthermore, we use the asymptotics for studying the capacity-achieving transmission strategies for the case of channel distribution information at the transmitter. Monte-Carlo simulations confirm the validity and accuracy of the analysis.

I. Introduction

A unifying solution to the moment generating function (MGF) of mutual information of arbitrarily correlated MIMO channels in a Rayleigh fading environment was recently presented by the authors in [10] for the standard channel model with Kronecker product covariance structure. Based on the MGF, closed-form ergodic capacity (EC) expressions were given in [11] and [12], thereby unifying existing solutions for uncorrelated and semi-correlated channels [1]-[5]. However, due to the mathematical complexity of the resulting EC formulas, a concise quantification of the influence of the various system and channel parameters on EC is difficult. Therefore, in this paper we present exact EC asymptotics in the low and high SNR regime, which allow for a deeper insight in the effects of fading correlation at the antenna arrays of wireless MIMO links. Specifically, we explicitly quantify the EC loss due to fading correlation at high SNR for an arbitrary number of transmit and receive antennas and fading correlation at transmit as well as receive antenna array. At this point we emphasize that our analysis is non-asymptotic in the number of antenna elements, an assumption that was made e.g. in [6]. We prove that fading correlation has no influence on the asymptotic slope of the EC curve in the high SNR regime, which is determined by the minimum of transmit and receive antennas as in case of uncorrelated fading. Moreover, we demonstrate that without channel state information (CSI) at the transmitter, in the low SNR regime ergodic capacity is independent of the correlation properties of the channel, which contrasts the behavior of EC with channel distribution information at the transmitter (CDIT) (see also [13]). Based on the capacity asymptotics, we study the capacity-achieving transmission strategies for the low and high SNR regime with CDIT. Finally, we present simulation results that show a perfect match with the asymptotical analysis of this paper.

II. Signal and Channel Model

We consider a flat fading MIMO link modeled by

$$\boldsymbol{y} = \boldsymbol{HFs} + \boldsymbol{n}, \tag{1}$$

where **s** is the L×1 TX symbol vector, i.e. there are L independent subchannels (data streams). **F** is a $T \times L$ linear matrix transmit prefilter that maps the L subchannels on the T transmit antennas, whereas $L \le T$. In the presence of CDIT, **F** can be adjusted for maximizing mean mutual information of the MIMO link (see below), otherwise we can set $\boldsymbol{F} = \boldsymbol{I}$. **H** is the $R \times T$ MIMO channel matrix with correlated Rayleigh fading elements, **n** is the $R \times 1$ noise vector, and **y** is the $R \times 1$ receive vector (see Fig. 1). By R we denote the number of receive antennas.

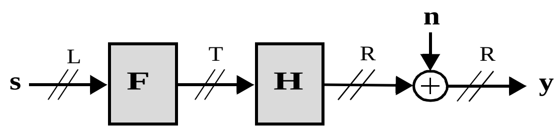

Fig. 1: System model

Without loss of generality (due to the presence of the transmit prefilter), we assume a white transmit symbol vector

$$\boldsymbol{R}_{ss} = E[\boldsymbol{ss}^H] = E_s \cdot \boldsymbol{I}_L, \tag{2}$$

where E_s is the mean energy per transmit symbol. Moreover, we always normalize the overall transmit power by setting $\mathrm{tr}(\boldsymbol{FF}^H) = \rho = T$, where ρ is a transmit power constraint. On the other hand, the noise covariance matrix reads

$$\boldsymbol{R}_{nn} = E[\boldsymbol{nn}^H] = N_0 \cdot \tilde{\boldsymbol{R}}_{nn} \tag{3}$$

with mean noise power per receive antenna N_0 and normalized matrix $\tilde{\boldsymbol{R}}_{nn}$ with $\mathrm{tr}(\tilde{\boldsymbol{R}}_{nn}) = R$.

Using a widely accepted channel model, the correlated MIMO channel can be described by the matrix product

$$\boldsymbol{H} = \boldsymbol{A}^H \boldsymbol{H}_w \boldsymbol{B}, \tag{4}$$

IEEE Vehicular Technology Conference (VTC), Milan, Italy, May 2004

where $\mathbf{H}_w$ is a $R \times T$ matrix of complex i.i.d. Gaussian variables of unity variance and

$$\boldsymbol{A}\boldsymbol{A}^H = \boldsymbol{R}_{RX} \qquad \boldsymbol{B}\boldsymbol{B}^H = \boldsymbol{R}_{TX}, \tag{5}$$

where $\mathbf{R}_{RX}$ and $\mathbf{R}_{TX}$ is the long-term stable normalized

$$\mathrm{tr}(\boldsymbol{R}_{RX}) = R \qquad \mathrm{tr}(\boldsymbol{R}_{TX}) = T \tag{6}$$

receive and transmit correlation matrix, respectively. Finally, we define the SNR in dB with $\gamma = E_s/N_0$

$$\gamma_{dB} = 10 \cdot \log_{10}(\rho \cdot \gamma), \tag{7}$$

In the remainder of the paper, by $\mathbf{I}_n$ we denote an identity matrix of size n×n (the index can be omitted, if the size of the matrix is clear from the context), diag(x_1,…x_n) is a diagonal matrix with elements x_1,…x_n, $|\boldsymbol{X}|$ is the determinant of the quadratic matrix $\mathbf{X}$, $[x_{ij}]$ is a matrix with element x_{ij} in row i and column j, $|x_{ij}|$ is the determinant of a matrix with element x_{ij} in row i and column j, eig($\mathbf{X}$) returns a diagonal matrix of eigenvalues of $\mathbf{X}$, $\mathbf{X}^H$ means Hermitian, $x \cong$ means 'random variable (RV) x has the same distribution as', and E_x[ ] denotes expected value with respect to RV x.

III. MIMO Mutual Information

A. General expressions

It is well known [1] that the mutual information $I(\boldsymbol{s}, \boldsymbol{y})$ between input vector **s** and output **y** of the equivalent MIMO channel in Fig. 1 is given by

$$I(\boldsymbol{s}, \boldsymbol{y}) = \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{F}\boldsymbol{F}^H\boldsymbol{H}^H\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{H}\right|. \tag{8}$$

Plugging the channel model (4) with Kronecker product covariance structure in (8), we find after some manipulations with the following definitions

$$\begin{aligned} \boldsymbol{O} &\equiv \mathrm{eig}(\tilde{\boldsymbol{R}}_{nn}^{-1}\boldsymbol{R}_{RX}) = \mathrm{diag}(o_1, \ldots, o_R) \\ \boldsymbol{S} &\equiv \mathrm{eig}(\boldsymbol{F}^H\boldsymbol{R}_{TX}\boldsymbol{F}) = \mathrm{diag}(s_1, \ldots, s_L) \end{aligned}, \tag{9}$$

the concise mutual information expression

$$I(\boldsymbol{s}, \boldsymbol{y}) \cong \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{S}\tilde{\boldsymbol{H}}_w^H\boldsymbol{O}\tilde{\boldsymbol{H}}_w\right| = \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{O}\tilde{\boldsymbol{H}}_w\boldsymbol{S}\tilde{\boldsymbol{H}}_w^H\right|, \tag{10}$$

where $\tilde{\boldsymbol{H}}_w$ is a R×L matrix of i.i.d. complex Gaussian elements. We rewrite (10) such that the matrix argument $\boldsymbol{S}\tilde{\boldsymbol{H}}_w^H\boldsymbol{O}\tilde{\boldsymbol{H}}_w$ or $\boldsymbol{O}\tilde{\boldsymbol{H}}_w\boldsymbol{S}\tilde{\boldsymbol{H}}_w^H$, respectively, of the determinant is of full rank, thereby simplifying the subsequent expressions. To this end, we define

$$\mu \equiv \min(R, L) \qquad \nu \equiv \max(R, L), \tag{11}$$

the $\mu \times \mu$ diagonal matrix

$$\boldsymbol{\Sigma} \equiv \begin{cases} \boldsymbol{S} & R \geq L \\ \boldsymbol{O} & L < R \end{cases} \qquad \boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_\mu), \tag{12}$$

the $\nu \times \nu$ diagonal matrix

$$\boldsymbol{\Omega} \equiv \begin{cases} \boldsymbol{O} & R \geq L \\ \boldsymbol{S} & L < R \end{cases} \qquad \boldsymbol{\Omega} = \mathrm{diag}(\omega_1, \ldots, \omega_\nu), \tag{13}$$

and the $\nu \times \mu$ matrix of i.i.d. complex Gaussian entries $\mathbf{G}$. With above definitions we can introduce a unifying expression for MIMO mutual information, which will serve as a basis for all following derivations

$$I(\boldsymbol{s}, \boldsymbol{y}) \cong \log_2\left|\boldsymbol{I} + \gamma \cdot \boldsymbol{\Sigma}\boldsymbol{G}^H\boldsymbol{\Omega}\boldsymbol{G}\right|. \tag{14}$$

IV. MIMO EC Asymptotics with Uninformed TX

The ergodic capacity C_{erg} with uninformed transmitter ($\boldsymbol{F} = \boldsymbol{I}$) is the first moment (expected value) of mutual information [1]

$$C_{erg,\boldsymbol{\Sigma\Omega}}(\gamma) = E_{\boldsymbol{G}}[I(\boldsymbol{s}, \boldsymbol{y})] = E_{\boldsymbol{G}}[\log_2|\boldsymbol{I} + \gamma \cdot \boldsymbol{\Sigma}\boldsymbol{G}^H\boldsymbol{\Omega}\boldsymbol{G}|]. \tag{15}$$

In the following subparagraphs, we explicitly determine the expected value in (15) with respect to the channel statistics for the high and low SNR regime.

A. High SNR regime

In [11] the exact expected value in (15) was explicitly calculated by using a MGF approach, which results in

Theorem 1. The exact ergodic capacity with transmit as well as receive correlation is given by

$$C_{erg,\boldsymbol{\Sigma\Omega}}(\gamma) = \frac{\Gamma_\nu(\nu) \cdot \gamma^{-\frac{\nu \cdot (\nu - 1)}{2}}}{\ln 2 \cdot \alpha_\mu(\boldsymbol{\Sigma}) \cdot \alpha_\nu(\boldsymbol{\Omega}) \cdot [\Gamma(\nu)]^\nu} \cdot \sum_{l=1}^{\mu} \begin{vmatrix} \Xi_{\boldsymbol{\Sigma\Omega}}(l) \\ \Psi_{\boldsymbol{\Sigma\Omega}}(0) \end{vmatrix} \tag{16}$$

with Gamma function $\Gamma(x)$, the definition

$$\Gamma_m(r) = \prod_{i=1}^{m} \Gamma(r - i + 1) \tag{17}$$

the $\nu \times \mu$ matrices (row index i runs from 1 to $\nu - \mu$ and column index j from 1 to ν)

$$\Xi_{\Sigma\Omega}(l) = \left[\begin{cases} \Gamma(\nu) \cdot \sigma_i^{\mu-1} \cdot (\gamma\omega_j)^{\nu-1} \cdot e^{\frac{1}{\gamma\sigma_i\omega_j}} \cdot \mathrm{E}_1\left(\frac{1}{\gamma\sigma_i\omega_j}\right) & i = l \\ \displaystyle\sum_{k=\nu-\mu}^{\nu-1} \sigma_i^{k-(\nu-\mu)} \cdot (\gamma\omega_j)^k \cdot \frac{\Gamma(\nu)}{\Gamma(\nu-k)} & i \neq l \end{cases}\right], \tag{18}$$

where $\mathrm{E}_1(x)$ is the exponential integral [7], and the $(\nu - \mu) \times \mu$ matrix (row index i' runs from 1 to μ and column index j' from 1 to ν)

$$\boldsymbol{\Psi}_{\boldsymbol{\Sigma\Omega}} = \left[(\gamma\omega_{j'})^{\nu-\mu-i'}\frac{\Gamma(\nu)}{\Gamma(\mu + i')}\right]. \tag{19}$$

Finally, $\alpha_m(\boldsymbol{X})$ denotes the Vandermonde determinant of the m×m diagonal matrix $\boldsymbol{X} = \mathrm{diag}(x_1, \ldots x_m)$.

Proof: See [10] and [11].

Via certain series expansions we can derive

Theorem 2. The ergodic capacity asymptotics $\bar{C}_{erg,\mathbf{\Sigma\Omega}}(\gamma_{dB})$ of a channel with correlation at TX and RX at high SNR with arbitrary number of transmit and receive antennas are given by

$$\bar{C}_{erg,\mathbf{\Sigma\Omega}}(\gamma_{dB}) = \begin{cases} \dfrac{-\mu\cdot(E+\ln\rho)+\ln|\mathbf{\Sigma}|+\zeta_1(\mathbf{\Omega})-\zeta_2(\mu,\nu)}{\ln 2} + \\ \mu\cdot\dfrac{\log_2 10}{10}\cdot\gamma_{dB} \end{cases} . \quad (20)$$

where $E \approx 0.5772156649$ is Euler's constant, $\zeta_1(\mathbf{\Omega})$ is an auxiliary function depending on $\mathbf{\Omega}$ (in the $\nu\times\nu$ determinants, i and j run from 1 to ν)

$$\zeta_1(\mathbf{\Omega}) = \frac{1}{\alpha_\nu(\mathbf{\Omega})}\cdot\sum_{l=1}^{\mu}\left|\begin{cases}\omega_j^{\nu-i}\cdot\ln\omega_j & i=l\\ \omega_j^{\nu-i} & i\neq l\end{cases}\right| \quad (21)$$

and the auxiliary function

$$\zeta_2(\mu,\nu) = \sum_{l=\nu-\mu}^{\nu-2} a_{E,\nu-1-l}\cdot\Gamma(\nu-l) \quad (22)$$

$$a_{E,k} = \sum_{l=1}^{k}\frac{(-1)^l}{l\cdot l!}\cdot\frac{1}{(k-l)!}. \quad (23)$$

Proof: See appendix for an outline.

Ergodic capacity is obviously linear in the SNR in dB (logarithmic in the SNR) at high SNR. We emphasize that the slope of the asymptotic capacity given in (20) is directly proportional to the minimum of the number of transmit and receive antennas μ, independent of the correlation properties of the channel. This agrees with results established via a large-dimensional random matrix analysis in [6].

We now consider some special cases of Theorem 2, which in principle can be directly derived from (20). However, we take a different approach for simplifying the derivation.

Corollary 1. The ergodic capacity asymptotics of a fully correlated $\nu\times\nu$ system are given by

$$\bar{C}_{erg,\mathbf{\Sigma\Omega},\nu\times\nu}(\gamma_{dB}) = \frac{-\nu\cdot\ln\rho+\ln(|\mathbf{\Sigma}||\mathbf{\Omega}|)+\sum_{k=0}^{\nu-1}\psi(\nu-k)}{\ln 2}+\nu\cdot\frac{\log_2 10}{10}\cdot\gamma_{dB}, \quad (24)$$

where $\psi(z)$ is the Digamma function [7, paragraph 6.3].

Proof: Note that at high SNR the mutual information in the $\nu\times\nu$ case can be approximated by

$$I(\mathbf{s},\mathbf{y})\approx\log_2|\gamma\cdot\mathbf{\Sigma}\mathbf{G}^H\mathbf{\Omega}\mathbf{G}| = \log_2|\mathbf{G}^H\mathbf{G}|+\log_2|\gamma\cdot\mathbf{\Sigma\Omega}|. \quad (25)$$

Now noting that $\mathbf{G}^H\mathbf{G}$ is complex i.i.d. Wishart $\tilde{\mathbf{W}}_\nu(\nu,\mathbf{I})$ distributed and using expected value

$$E_{\mathbf{W}}[\ln|\mathbf{W}|] = \sum_{k=0}^{m-1}\psi(n-k) \quad (26)$$

for Wishart $\tilde{\mathbf{W}}_m(n,\mathbf{I})$ distributed matrix **W**, which was also used in [9], with (7) we directly get (24). *QED.*

The following corollary can be obtained similarly and thus we state without proof

Corollary 2. The ergodic capacity high SNR asymptotics of a MIMO system with semi-correlated channel and $\mathbf{\Omega} = \mathbf{I}$ are given by

$$\bar{C}_{erg,\mathbf{\Sigma}}(\gamma_{dB}) = \frac{1}{\ln 2}\cdot\left(-\mu\cdot\ln\rho+\ln(|\mathbf{\Sigma}|)+\sum_{k=0}^{\mu-1}\psi(\nu-k)\right)+\mu\cdot\frac{\log_2 10}{10}\cdot\gamma_{dB}. \quad (27)$$

It directly follows from (20) and (27)

Corollary 3. The asymptotic EC loss $\Delta\bar{C}_{erg,\mathbf{\Omega}}$ in bit per channel use due to correlation $\mathbf{\Omega}\neq\mathbf{I}$ is given by

$$\Delta\bar{C}_{erg,\mathbf{\Omega}} = \frac{\left(\sum_{k=0}^{\mu-1}\psi(\nu-k)\right)+\mu\cdot E-\zeta_1(\mathbf{\Omega})+\zeta_2(\mu,\nu)}{\ln 2}. \quad (28)$$

The asymptotic loss due to $\mathbf{\Sigma}\neq\mathbf{I}$ simply reads

$$\Delta\bar{C}_{erg,\mathbf{\Sigma}} = -\log_2|\mathbf{\Sigma}|. \quad (29)$$

The negative impact of the correlation matrix $\mathbf{\Sigma}$ at that side of the MIMO link with less antennas is directly obvious in (29). Specifically, the determinant of a matrix is a Schur-concave function of the eigenvalues (see [8]), i.e. as expected higher correlation reduces capacity.

B. Low SNR regime

Theorem 3. The ergodic capacity $\underline{C}_{erg}(\gamma)$ of an arbitrarily correlated MIMO channel in the low SNR region is given by

$$\underline{C}_{erg}(\gamma) = \frac{\mathrm{tr}(\mathbf{\Sigma})\cdot\mathrm{tr}(\mathbf{\Omega})}{\ln 2}\cdot\gamma. \quad (30)$$

With the typical normalization of the correlation matrices in (6), $\mathbf{F} = \mathbf{I}$, and AWGN, we find with $\rho = T$

$$\underline{C}_{erg}(\gamma) = \frac{R\cdot T}{\ln 2}\cdot\gamma = \frac{R\cdot\rho}{\ln 2}\cdot\gamma = \frac{R\cdot\rho}{\ln 2}\cdot\frac{1}{\rho}\cdot 10^{\frac{\gamma_{dB}}{10}} = \frac{R}{\ln 2}\cdot 10^{\frac{\gamma_{dB}}{10}}. \quad (31)$$

Proof: In principle, the derivation can again be based on the exact capacity formulas in Theorem 1. However, the derivation is tedious and therefore, we present a concise alternative proof. To this end, consider the formula

$$\ln|\mathbf{I}-x\mathbf{A}| = -\sum_{k=1}^{\infty}\frac{1}{k}\cdot x^k\cdot\mathrm{tr}(\mathbf{A}^k) \quad (32)$$

for scalar x and matrix $\mathbf{A}$ with $\|x\mathbf{A}\| < 1$. For small x we obviously get

$$\ln|\boldsymbol{I} - x\boldsymbol{A}| \approx -x \cdot \mathrm{tr}(\boldsymbol{A}) \,. \tag{33}$$

Then observe the ergodic capacity expression (14). For low SNR we arrive with (33) and $E_{\boldsymbol{G}}[\boldsymbol{G}^H \boldsymbol{\Omega} \boldsymbol{G}] = \mathrm{tr}(\boldsymbol{\Omega}) \cdot \boldsymbol{I}_\mu$ at (30). *QED.*

Obviously, in the low SNR regime, ergodic capacity is linear in the SNR (or exponential in the SNR in dB) and with AWGN it is independent of fading correlation.

V. MIMO EC Asymptotics with CDIT

With the availability of channel distribution information at the transmitter (CDIT), it is well-known that the capacity-achieving transmission strategy is given by

$$\boldsymbol{F} = \tilde{\boldsymbol{V}}_{TX} \cdot \boldsymbol{\Phi} \tag{34}$$

with diagonal power allocation (PA) matrix $\boldsymbol{\Phi} = \mathrm{diag}(\phi_1, \ldots, \phi_L)$ and the matrix of eigenvectors $\tilde{\boldsymbol{V}}_{TX}$ corresponding to the L largest eigenvalues of the transmit correlation matrix $\tilde{\boldsymbol{\Lambda}}_{TX} = \mathrm{diag}(\lambda_{TX,1}, \ldots, \lambda_{TX,L})$. This results in (see (9))

$$\boldsymbol{S}(\boldsymbol{\Phi}) = \mathrm{eig}(\tilde{\boldsymbol{\Lambda}}_{TX} \boldsymbol{\Phi}^2) = \mathrm{diag}(\lambda_{TX,1}\phi_1^2, \ldots, \lambda_{TX,L}\phi_L^2) \,. \tag{35}$$

With CDIT, ergodic MIMO capacity is then given by

$$C_{\mathrm{erg}}^{CDIT}(\gamma) = \max_{\boldsymbol{\Phi}} E[I(\boldsymbol{s}, \boldsymbol{y}, \boldsymbol{\Phi})] \qquad \text{s.t. } \mathrm{tr}(\boldsymbol{\Phi}\boldsymbol{\Phi}^H) = \rho \,. \tag{36}$$

If $R \geq T$, it follows from above analysis that in the high SNR regime power should be equally distributed over all T long-term eigenmodes of the channel with L=T, such that the EC curve achieves full slope. Furthermore, it follows from Corollary 3 with the Schur-concavity of the determinant of $\boldsymbol{\Phi}$ that the optimum PA strategy is $\boldsymbol{\Phi} = \boldsymbol{I}_L$. Obviously, for this special case, there is no statistical waterfilling gain. On the other hand, if R<T, we have to set $L \geq R$ to achieve full slope of EC at high SNR and the optimal PA matrix has to be determined numerically [13]. In this scenario, the beamforming-like capabilities of the transmit antenna array can be exploited and a statistical waterfilling gain can be achieved.

In the low SNR regime, it is a straightforward exercise to show that the optimum capacity-achieving PA strategy is given by $\boldsymbol{\Phi} = \mathrm{diag}(\sqrt{\rho}, 0, \ldots, 0)$, such that all transmission power is assigned to the strongest long-term eigenmode of the channel, which maximizes (30).

VI. Simulation Results

Without loss of generality, we study a system with additive white Gaussian noise of variance N_0, i.e. $\boldsymbol{R}_{nn} = N_0 \cdot \boldsymbol{I}$. In the following, we consider exponential correlation matrices at the receiver and the transmitter with

$$[\boldsymbol{R}_{\mathrm{RX/TX}}]_{ij} = (r_{\mathrm{RX/TX}})^{|i-j|} \,, \tag{37}$$

i.e. r_{RX} is the correlation coefficient between two neigboring receive antennas and r_{TX} models the correlation between two transmit antennas. Moreover, the correlation between two antennas decreases exponentially with their distance. The transmitter is assumed to be uninformed, i.e. it has no CSI and L=T. Simulation results and theoretical asymptotics closely agree in Fig. 2 for a system with T=4 transmit and R=6 receive antennas. The correlation at the receiver is kept constant r_{RX}=0.7, while the correlation at transmitter increases from left to right.

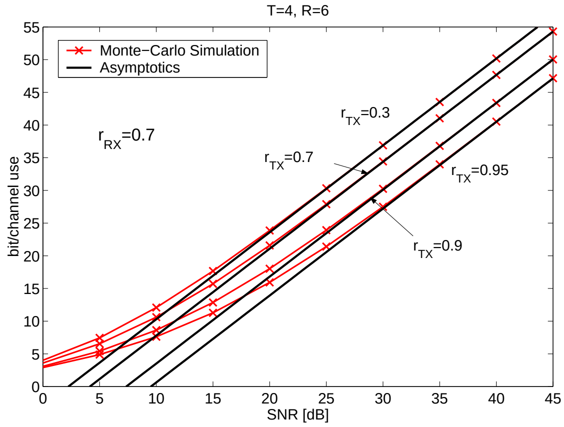

Fig. 2: EC asymptotics, T=L=4, R=6, r_{TX}={0.3,0.7,0.9,0.95}, r_{RX}=0.7

Analytical asymptotics of ergodic capacity and simulation results are depicted on a logarithmic scale in Fig. 3 for the low SNR regime. Again, the receive correlation is fixed r_{RX}=0.7 and we vary the transmit correlation for a system with T=3 and R=5. As predicted by the theoretical analysis, EC is invariant to fading correlation in the low SNR regime, while the curves significantly deviate for SNR values higher than -10 dB. Moreover, clearly the linear increase of EC in SNR changes to a logarithmic increase.

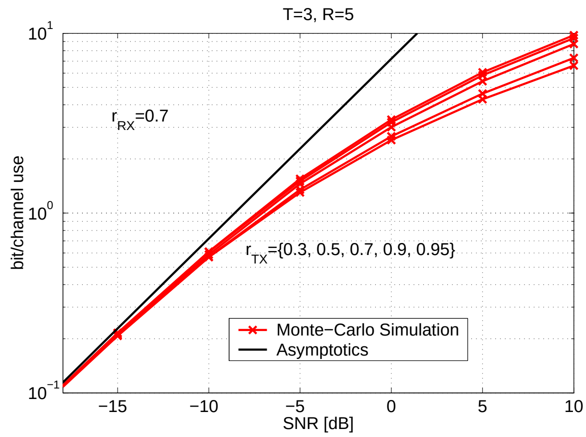

Fig. 3: EC, low SNR, L=T=3, R=5, r_{TX}={0.3,0.5,0.7,0.9,0.95}, r_{RX}=0.7

Appendix

With the help of the series expansion of the exponential integral E_1 (see [7]) it is possible to show that

$$e^{1/z}\cdot E_1\left(\frac{1}{z}\right) = (-E+\ln z)\cdot\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}-\sum_{k=1}^{\infty}a_{E,k}\cdot z^{-k} \quad (38)$$

with $a_{E,k}$ in (23). Furthermore, for column vectors **a**, **b**, and matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ of compatible size it can be shown that the following determinant expansion holds

$$\begin{vmatrix} \boldsymbol{C}_1 \\ \boldsymbol{a}^T+\boldsymbol{b}^T \\ \boldsymbol{C}_2 \end{vmatrix} = \begin{vmatrix} \boldsymbol{C}_1 \\ \boldsymbol{a}^T \\ \boldsymbol{C}_2 \end{vmatrix} + \begin{vmatrix} \boldsymbol{C}_1 \\ \boldsymbol{b}^T \\ \boldsymbol{C}_2 \end{vmatrix}. \quad (39)$$

With the help of (38) and (39) we can split the determinants in (16)

$$\begin{vmatrix} \Xi_{\Sigma\Omega}(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} = \begin{vmatrix} \boldsymbol{U}_1(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} + \begin{vmatrix} \boldsymbol{U}_2(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} - \begin{vmatrix} \boldsymbol{U}_3(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix}, \quad (40)$$

with the $\mu\times\nu$ auxiliary matrices (i runs from 1 to μ and j from 1 to ν)

$$\boldsymbol{U}_1(l) = \left[\begin{cases}\Gamma(\nu)\cdot\sigma_i^{\mu-1}\cdot(\gamma\omega_j)^{\nu-1}\cdot(-E+\ln\gamma\sigma_i)\cdot\sum_{n=0}^{\infty}\frac{(\gamma\sigma_i\omega_j)^{-n}}{n!} & i=l \\ \sum_{k=\nu-\mu}^{\nu-1}\sigma_i^{k-(\nu-\mu)}\cdot(\gamma\omega_j)^k\cdot\frac{\Gamma(\nu)}{\Gamma(\nu-k)} & i\neq l\end{cases}\right], \quad (41)$$

$$\boldsymbol{U}_2(l) = \left[\begin{cases}\Gamma(\nu)\cdot\sigma_i^{\mu-1}\cdot(\gamma\omega_j)^{\nu-1}\cdot\ln\omega_j\cdot\sum_{n=0}^{\infty}\frac{(\gamma\sigma_i\omega_j)^{-n}}{n!} & i=l \\ \sum_{k=\nu-\mu}^{\nu-1}\sigma_i^{k-(\nu-\mu)}\cdot(\gamma\omega_j)^k\cdot\frac{\Gamma(\nu)}{\Gamma(\nu-k)} & i\neq l\end{cases}\right], \quad (42)$$

and finally

$$\boldsymbol{U}_3(l) = \left[\begin{cases}\Gamma(\nu)\cdot\sigma_i^{\mu-1}\cdot(\gamma\omega_j)^{\nu-1}\cdot\sum_{k=1}^{\infty}a_{E,k}\cdot(\gamma\sigma_i\omega_j)^{-k} & i=l \\ \sum_{k=\nu-\mu}^{\nu-1}\sigma_i^{k-(\nu-\mu)}\cdot(\gamma\omega_j)^k\cdot\frac{\Gamma(\nu)}{\Gamma(\nu-k)} & i\neq l\end{cases}\right]. \quad (43)$$

By elementary row operations and neglecting negative powers of γ at high SNR, we can derive from (41)

$$\begin{vmatrix} \boldsymbol{U}_1(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} \approx \begin{vmatrix} \begin{cases}(-E+\ln\gamma\sigma_i)\cdot\sum_{k=\nu-\mu}^{\nu-1}\sigma_i^{k-(\nu-\mu)}\cdot(\gamma\omega_j)^k\cdot\frac{\Gamma(\nu)}{\Gamma(\nu-k)} & i=l \\ \sum_{k=\nu-\mu}^{\nu-1}\sigma_i^{k-(\nu-\mu)}\cdot(\gamma\omega_j)^k\cdot\frac{\Gamma(\nu)}{\Gamma(\nu-k)} & i\neq l\end{cases} \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix}. \quad (44)$$

After expanding the determinant via (39) we can row-wise extract common factors and find after simplifying

$$\begin{vmatrix} \boldsymbol{U}_1(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} \approx \gamma^{\frac{\nu\cdot(\nu-1)}{2}}\cdot\frac{[\Gamma(\nu)]^{\nu}}{\Gamma_{\nu}(\nu)}\cdot(-E+\ln\gamma\sigma_l)\cdot\begin{vmatrix} \sum_{k=\nu-\mu}^{\nu-1}\sigma_i^{k-(\nu-\mu)}\cdot\omega_j^k \\ \boldsymbol{V} \end{vmatrix} \quad (45)$$

with the $(\nu-\mu)\times\nu$ matrix (i' runs from 1 to $\nu-\mu$ and j' from 1 to ν)

$$\boldsymbol{V} = [\omega_{j'}^{\nu-\mu-i'}]. \quad (46)$$

It can then be shown that (45) can be written in terms of a product of Vandermonde determinants

$$\begin{vmatrix} \boldsymbol{U}_1(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} \approx \gamma^{\frac{\nu\cdot(\nu-1)}{2}}\cdot\frac{[\Gamma(\nu)]^{\nu}}{\Gamma_{\nu}(\nu)}\cdot(-E+\ln\gamma\sigma_l)\cdot\alpha_{\mu}(\boldsymbol{\Sigma})\cdot\alpha_{\nu}(\boldsymbol{\Omega}). \quad (47)$$

Similarly, we can find

$$\begin{vmatrix} \boldsymbol{U}_2(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} \approx \gamma^{\frac{\nu\cdot(\nu-1)}{2}}\cdot\frac{[\Gamma(\nu)]^{\nu}}{\Gamma_{\nu}(\nu)}\cdot\alpha_{\mu}(\boldsymbol{\Sigma})\cdot\begin{vmatrix}\begin{cases}\omega_j^{\nu-i}\cdot\ln\omega_j & i=l \\ \omega_j^{\nu-i} & i\neq l\end{cases}\end{vmatrix} \quad (48)$$

and finally

$$\sum_{l=1}^{\mu}\begin{vmatrix} \boldsymbol{U}_3(l) \\ \boldsymbol{\Psi}_{\Sigma\Omega} \end{vmatrix} = \frac{[\Gamma(\nu)]^{\nu}\cdot\alpha_{\mu}(\boldsymbol{\Sigma})\cdot\alpha_{\nu}(\boldsymbol{\Omega})}{\Gamma_{\nu}(\nu)}\cdot\sum_{l=\nu-\mu}^{\nu-2}a_{E,\nu-1-l}\cdot\Gamma(\nu-l) \quad . \quad (49)$$

Combining and simplifying the partial results (47)-(49) yields finally Theorem 2. *QED.*

REFERENCES

[1] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Bell Labs Technical Memorandum*, June 1995

[2] Z. Wang, G. B. Giannakis, "Outage mutual information of space-time MIMO channels," *IEEE Transactions on Information Theory*, 2003, *submitted*

[3] Chiani M., Win M. Z., Zanella A., "On the capacity of spatially correlated MIMO Rayleigh-fading channels," *IEEE Transactions on Information Theory*, vol. 49, no. 10, Oct. 2003

[4] Smith P. J., Roy S., Shafi M., "Capacity of MIMO systems with semicorrelated flat fading," *IEEE Transactions on Information Theory*, vol. 49, no. 10, Oct. 2003

[5] Kang M., Alouini M.- S., "On the outage capacity of MIMO channels," *IEEE Transactions on Information Theory*, 2003, *submitted*

[6] Chuah C.-N., Tse D. N. C., Kahn J. M., Valenzuela R. A., "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Transactions on Information Theory,* vol. 48, no.3, pp. 637-650, Mar. 2002

[7] Abramowitz M., Stegun I. A., *Handbook of mathematical functions*, Dover Publications, Inc., New York 1964

[8] Marshall A. W., Olkin I., *Inequalities: Theory of majorization and its applications*, Academic Press, 1979

[9] Grant A., "Rayleigh fading multiple-antenna channels," EURASIP Journal on Applied Signal Processing, Special Issue on Space-Time Coding (Part I), vol 2002, no. 3, pp. 316-329, March 2002

[10] Kiessling M., Speidel J., "Mutual information of MIMO channels in correlated Rayleigh fading environments - a general solution", ICC, May 2004

[11] Kiessling M., Speidel J., "Exact ergodic capacity of MIMO channels in correlated Rayleigh fading environments," International Zurich Seminar on Communications, Feb. 2004

[12] Kiessling M., Speidel J., Reinhardt M., "Unifying analysis of ergodic MIMO capacity in correlated Rayleigh fading environments," *European Wireless Conference,* Feb. 2004

[13] Kiessling M., Speidel J., Reinhardt M., "Ergodic capacity of MIMO channels with statistical channel state information at the transmitter," *ITG Workshop on Smart Antennas,* March 2004