Exact Symbol Error Probability of M-PSK for Multihop Transmission with Regenerative Relays

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***Abstract*— We determine the exact symbol error probability of M-ary phase shift keying (M-PSK) for multihop communication systems with regenerative relays, where the source terminal transmits data to the destination terminal via a set of intermediate relay stations, which perform hard decisions on the received symbols before forwarding them to their respective successor node. Both, time-invariant additive white Gaussian noise channels as well as frequency-flat fading channels are considered and we derive generic expressions, which might be easily evaluated numerically or even be given in closed-form for various cases.**

***Index Terms*— Multihop transmission, symbol error probability, M-PSK, relay channel, decode-and-forward, Nakagami-m fading, regenerative relays.**

I. Introduction

MULTIHOP transmission represents a very attractive and promising way for improving the performance and efficiency of future wireless communication systems [1], [2]. With multihop transmission, the source generally transmits its data not directly to the desired destination, but rather on a hop-by-hop basis via various intermediate relay nodes. This might entail a number of different benefits: In cellular networks, for example, the cell capacity and cell coverage can be improved, without having to increase the transmit power of the base stations/mobile terminals and hence risking to violate imposed power constraints [1]. Besides, relayed transmission represents a key enabling technology for wireless ad-hoc and sensor networks, which usually cannot revert to any fixed network infrastructure. By appropriately designing multihop systems, it is also possible to improve the power efficiency of a system, i.e., the total power consumed by all participating nodes might be smaller than the power needed for a conventional single-hop system without impairing the system performance [3].

The design and performance analysis of multihop communication systems has gained a lot of research attention in recent years, but most studies have focused on the outage probability and average bit error rate so far, see for example [4]–[6]. In this letter—in order to facilitate a more profound analysis and optimization of such systems—we rather determine the exact average symbol error probability (SEP) of relayed transmission with M-ary phase shift keying (M-PSK) for a general multihop system with an arbitrary number of regenerative relays, where all relays perform hard decisions on the corresponding received symbols before forwarding them to the next node.

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II. SEP Analysis for AWGN Channels

A. Dual-Hop Transmission

First of all, we consider a simple dual-hop system consisting of the actual source and destination terminals as well as an intermediate relay station. Initially, the source transmits a certain data symbol s to the relay, which in case of additive white Gaussian noise (AWGN) receives $y_R = s + n_R$, where n_R denotes a complex Gaussian random variable with zero mean and variance σ_R^2, i.e., $n_R \sim \mathcal{CN}(0, \sigma_R^2)$. The relay then estimates the transmitted symbol s and forwards its estimate $\hat{s}$ to the destination node, which receives $y_D = \hat{s} + n_D$ with $n_D \sim \mathcal{CN}(0, \sigma_D^2)$. For notational convenience, we assume without loss of generality that the transmit symbol energy is normalized to one and we set $\sigma_R^2 = \frac{1}{\gamma_R}$ and $\sigma_D^2 = \frac{1}{\gamma_D}$, where γ_R and γ_D denote the average signal-to-noise ratios (SNR) on the source-to-relay and relay-to-destination link, respectively.

With dual-hop transmission, the overall transmission might be error-free even if the hard decision made by the relay was incorrect, namely if the destination performs at the same time another erroneous decision on the symbol received from the relay which compensates the first error. Denoting the different symbols of the considered M-PSK constellation by $s_1, \ldots, s_M$, it can easily be shown that the average probability P_c of correct end-to-end transmission is generally given by

$$P_c = \sum_{i=1}^{M} P[s_i] \sum_{k=1}^{M} P\left[s_i + n_R \in D_k\right] P\left[s_k + n_D \in D_i\right], \quad (1)$$

where $P[s_i]$ denotes the probability that s_i is transmitted and D_i corresponds to the decision region associated with s_i. Hence, $P[s_i + n_R \in D_k]$ is simply the probability that the relay decides in favor of $\hat{s} = s_k$ when s_i has been transmitted and, accordingly, $P\left[s_k + n_D \in D_i\right]$ is the probability that the destination decides in favor of s_i when the relay has forwarded s_k. Assuming equiprobable transmit symbols, i.e., $P[s_i] = \frac{1}{M}\ \forall\ i$, as well as a regular signal point constellation with equidimensional decision regions D_i, (1) simplifies to

$$P_c = \sum_{k=1}^{M} P\left[s_1 + n_R \in D_k\right]\ P\left[s_k + n_D \in D_1\right]. \quad (2)$$

Systems with unequal transmission probabilities $P[s_i]$ or non-uniform decision regions can be treated in a straightforward manner similar to the analysis presented in the following, but since the resulting expressions quickly become rather lengthy, we do not consider this generalization in more detail here.

For calculating (2), we first of all consider the probability

$$P_k(\gamma) = P[s_1 + n \in D_k], \quad k = 1, \ldots, M \quad (3)$$

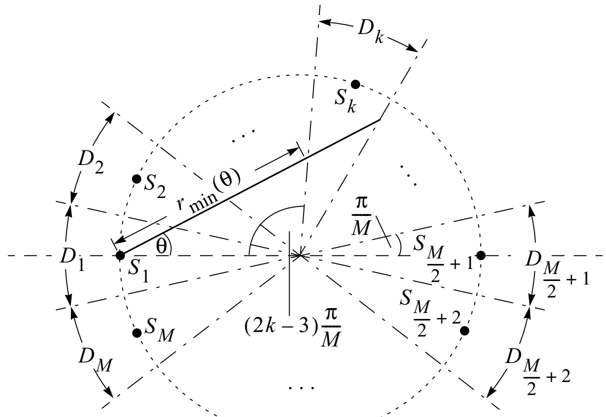

Fig. 1. General M-PSK signal point constellation with decision regions.

where $n \sim \mathcal{CN}(0, 1/\gamma)$. In polar coordinates, the probability density function (pdf) of n is well-known to be given by

$$p_n(r,\theta) = \frac{1}{\pi}\gamma\, r\, e^{-\gamma\, r^2}, \quad r \geq 0, \quad 0 \leq \theta < 2\pi \tag{4}$$

where $r = |n|$ and $\theta = \arg\{n\}$. A first question to be answered is for which values of r the received symbol $s_1 + n$ falls into D_k given a certain value of θ. From Fig. 1, it can easily be seen that for $k = 2, \ldots, \frac{M}{2}$, $s_1 + n$ may only lie within D_k if $0 \leq \theta < (2k-3)\pi/M$. If this condition is fulfilled, it can be shown with the law of sines—similarly to the analysis in [7]—that the minimum required value for r is given by

$$r_{\text{min}} = \frac{\sin\vartheta_{k-1}}{\sin(\pi - \theta - \vartheta_{k-1})} \tag{5}$$

while the maximum allowed value can be determined as

$$r_{\text{max}} = \begin{cases} \frac{\sin\vartheta_k}{\sin(\pi-\theta-\vartheta_k)} & \text{if } 0 \leq \theta \leq \pi - \vartheta_k \\ \infty & \text{if } \pi - \vartheta_k \leq \theta < \pi - \vartheta_{k-1} \end{cases}, \tag{6}$$

where we have introduced for brevity the short-hand notation

$$\vartheta_k = (2k-1)\frac{\pi}{M}. \tag{7}$$

Consequently, we obtain for $k = 2, 3, \ldots, \frac{M}{2}$

$$P_k(\gamma) = \int_0^{\pi-\vartheta_k} \int_{\frac{\sin\vartheta_{k-1}}{\sin(\pi-\theta-\vartheta_{k-1})}}^{\frac{\sin\vartheta_k}{\sin(\pi-\theta-\vartheta_k)}} p_n(r,\theta)dr\, d\theta + \int_{\pi-\vartheta_k}^{\pi-\vartheta_{k-1}} \int_{\frac{\sin\vartheta_{k-1}}{\sin(\pi-\theta-\vartheta_{k-1})}}^{\infty} p_n(r,\theta)dr\, d\theta, \tag{8}$$

which simplifies after solving the inner integrals with (4) to

$$P_k(\gamma) = \frac{1}{2\pi}\left[\int_0^{\pi-\vartheta_{k-1}} e^{\gamma\frac{\sin^2\vartheta_{k-1}}{-\sin^2 y}}dy - \int_0^{\pi-\vartheta_k} e^{\gamma\frac{\sin^2\vartheta_k}{-\sin^2 y}}dy\right] \tag{9}$$

Due to the assumed symmetry of the considered signal point constellations, it is quite obvious that generally

$$P_{M-k+2}(\gamma) = P_k(\gamma), \quad 2 \leq k \leq \frac{M}{2} \tag{10}$$

as well as

$$P[s_1 + n \in D_k] = P[s_k + n \in D_1], \quad 1 \leq k \leq M. \tag{11}$$

Therefore, the only missing terms that are needed for evaluating (2) are $P_1(\gamma)$ as well as $P_{\frac{M}{2}+1}(\gamma)$, which can similarly to the previous analysis be shown to be given by

$$P_{\frac{M}{2}+1}(\gamma) = \frac{1}{\pi}\int_0^{\pi-\frac{\pi}{M}} e^{-\gamma\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2 y}}dy \tag{12}$$

$$P_1(\gamma) = 1 - \frac{1}{\pi}\int_0^{\pi-\frac{\pi}{M}} e^{-\gamma\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2 y}}dy. \tag{13}$$

Based on (2), the exact average symbol error probability in AWGN hence can be calculated as

$$P_e = 1 - P_c = 1 - \sum_{k=1}^{M} P_k(\gamma_R)P_k(\gamma_D), \tag{14}$$

with $P_k(\gamma)$ according to (9)–(13). Similarly to the standard SEP expression for M-PSK in single-hop systems derived in [7], (14) might be easily evaluated numerically since this involves only the calculation of integrals with finite integration limits and integrands. Besides, it can easily be checked that

$$\lim_{\gamma_D\to\infty} P_e(\gamma_R, \gamma_D) = \frac{1}{\pi}\int_0^{\pi-\frac{\pi}{M}} e^{-\gamma_R\frac{\sin^2\left(\frac{\pi}{M}\right)}{\sin^2 y}}dy, \tag{15}$$

which is simply the SEP of the first hop [7]. This is reasonable because the second hop is error-free in that case. Likewise, if $\gamma_R \to \infty$, P_e corresponds to the SEP of the second hop.

B. Extension to an Arbitrary Number of Hops

In the more general case with N consecutive relays, the probability P_c of correct end-to-end transmission basically can be determined in the same way as before, namely by considering all possible combinations of correct and erroneous decisions at the relays and the destination for which the end-to-end transmission itself is error-free. It can easily be seen that this probability consequently is generally given by

$$P_c = \sum_{k_0,\ldots,k_N} P[s_{k_0}]\, P[s_{k_0} + n_0 \in D_{k_1}]P[s_{k_1} + n_1 \in D_{k_2}] \ldots P[s_{k_{N-1}} + n_{N-1} \in D_{k_N}]P[s_{k_N} + n_N \in D_{k_0}], \tag{16}$$

where $n_i \sim \mathcal{CN}(0, 1/\gamma_i)$ with γ_i as the SNR on the i-th hop and where the summation has to be taken over all possible index tuples $(k_0, \ldots, k_N)$ with $k_i = 1, \ldots, M$ $(i = 0, 1, \ldots, N)$. As before, we assume for simplicity equiprobable a priori probabilities and equidimensional decision regions, in which case (16) might be rewritten in a more compact form as

$$P_c = \frac{1}{M}\sum_{k_0,\ldots,k_N} P[s_{k_N}+n_N \in D_{k_0}]\prod_{i=0}^{N-1} P[s_{k_i}+n_i \in D_{k_{i+1}}]. \tag{17}$$

Furthermore, it can easily be seen that generally

$$P[s_i + n_\nu \in D_j] = P_{|j-i|+1}(\gamma_\nu), \tag{18}$$

with $P_k(\gamma)$ according to (9)–(13). The exact average symbol error probability $P_e = 1 - P_c$ for the general multihop case hence can be expressed in a generic form as

$$P_e = 1 - \frac{1}{M}\sum_{k_0,\ldots,k_N} P_{|k_0-k_N|+1}(\gamma_N)\prod_{i=0}^{N-1} P_{|k_{i+1}-k_i|+1}(\gamma_i), \tag{19}$$

which again might be easily evaluated numerically.

III. Average SEP in Fading Channels

The average SEP in frequency-flat fading channels generally can be obtained by averaging (19) over the distributions of the SNRs γ_i $(i = 0, \ldots, N)$ of the individual hops, i.e.,

$$\bar{P}_e = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(N+1)-\text{fold}} P_e(\gamma_0, \gamma_1, \ldots, \gamma_N) \prod_{i=0}^{N} p_{\gamma_i}(\gamma_i) d\gamma_i, \quad (20)$$

where $p_{\gamma_i}(\gamma_i)$ denotes the pdf of γ_i. Here, we implicitly assumed that all γ_i are independent of each other, but this should be a reasonable assumption if the distance between the individual nodes is large enough, what is expected to be the case in most scenarios of practical interest. After inserting (19) together with (9)–(13) in (20), separating the integrals and changing the order of integration, we obtain a sometimes more beneficial form based on the moment-generating functions $M_{\gamma_i}(s)$ of the link SNRs γ_i, which is given by

$$\bar{P}_e = 1 - \frac{1}{M} \sum_{k_0, \ldots, k_N} \Phi_{|k_0 - k_N|+1}(\gamma_N) \prod_{i=0}^{N-1} \Phi_{|k_{i+1} - k_i|+1}(\gamma_i), \quad (21)$$

where

$$\Phi_k(\gamma_i) = \begin{cases} 1 - \frac{1}{\pi} \int_0^{\pi - \frac{\pi}{M}} M_{\gamma_i}\left(\frac{\sin^2 \frac{\pi}{M}}{-\sin^2 y}\right) dy, & k = 1 \\ \frac{1}{\pi} \int_0^{\pi - \frac{\pi}{M}} M_{\gamma_i}\left(\frac{\sin^2 \frac{\pi}{M}}{-\sin^2 y}\right) dy, & k = \frac{M}{2} + 1 \\ \frac{1}{2\pi} \Big[\int_0^{\pi - \vartheta_{k-1}} M_{\gamma_i}\left(\frac{\sin^2 \vartheta_{k-1}}{-\sin^2 y}\right) dy & \\ \quad - \int_0^{\pi - \vartheta_k} M_{\gamma_i}\left(\frac{\sin^2 \vartheta_k}{-\sin^2 y}\right) dy \Big] & \text{else} \end{cases} \quad (22)$$

with ϑ_k according to (7). This generic expression for the average SEP might be easily evaluated numerically for a wide variety of different fading distributions since the calculation involves only integrals with finite integration limits again. Furthermore, in many cases closed-form solutions can be derived. For the important case of Nakagami-m fading with (integer) fading parameter m_i and average SNR $\bar{\gamma}_i$ on the i-th hop, for example, we have [8]

$$M_{\gamma_i}(s) = \left(1 - \frac{s\bar{\gamma}_i}{m_i}\right)^{-m_i}, \quad i = 0, 1, \ldots, N. \quad (23)$$

Inserting (23) in (22) and capitalizing on [8] eq. (5A.35), $\Phi_k(\gamma_i)$ and hence $\bar{P}_e$ can be given in closed-form, what, however, is not explicitly shown here due to space constraints.

IV. Numerical Examples

Fig. 2 shows the average SEP of a dual-hop system versus the average SNR of the source-to-relay link in AWGN for different values of γ_D. As can be seen, if the source-to-relay link is rather good, the average SEP does not further decrease with increasing values of γ_R. This is because for $\gamma_R \to \infty$ the overall SEP corresponds to the SEP of the relay-to-destination link which is independent of γ_R, as already outlined before.

Fig. 3 depicts the impact of the fading severity on the average SEP of a dual-hop system for QPSK and Nakagami-m fading with $\bar{\gamma}_R$ = $\bar{\gamma}_D$ = 15 dB. Clearly, the fading severity has also a significant impact on the performance and for $m_R = 20$, for instance, the average SEP decreases in the order of about four magnitudes if m_D is increased from one to 20.

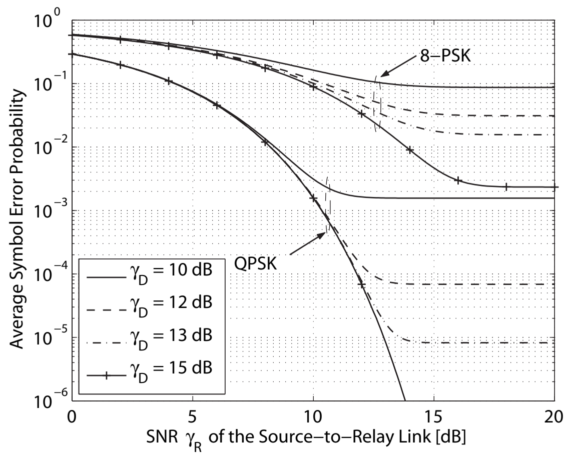

Fig. 2. Average SEP of a dual-hop system in AWGN as a function of the SNR γ_R on the source-to-relay link for QPSK and 8-PSK modulation and several different SNRs γ_D on the relay-to-destination link.

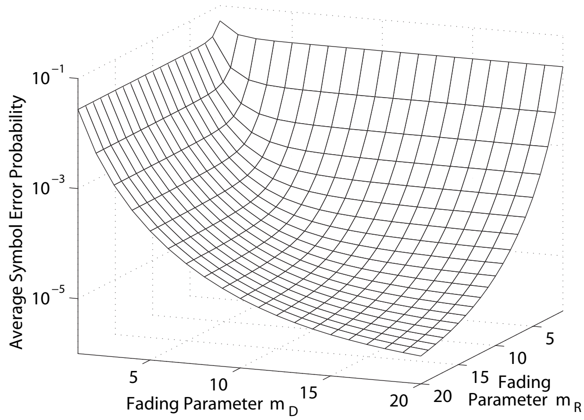

Fig. 3. Impact of the fading severity on the source-to-relay link (m_R) and the relay-to-destination link (m_D) on the average SEP of a dual-hop system with QPSK modulation in Nakagami-m fading with $\bar{\gamma}_D = \bar{\gamma}_R = 15$ dB.

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