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Characterization of Mutual Information of Spatially Correlated MIMO Channels with Keyhole

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***Abstract*— We characterize the statistical properties of the mutual information between the transmitter and the receiver of a multiple-input multiple-output (MIMO) communication system in case of spatially correlated Rayleigh-fading keyhole channels with arbitrarily correlated input signals and potentially colored additive Gaussian noise. The probability density function (pdf) as well as the cumulative distribution function (cdf) of the mutual information are derived in a concise mathematical form and we determine exact analytical closed-form expressions for its mean value as well as the corresponding high SNR asymptotics. Numerical results illustrate the impact of several different parameters on the mutual information statistics and are shown to be in perfect agreement with results obtained from Monte-Carlo simulations, thus verifying the accuracy of our theoretical analysis.**

I. Introduction

Multiple-input multiple-output (MIMO) systems are known to offer a wide variety of benefits compared to conventional single-input single-output (SISO) systems, such as the potential to realize considerably higher data rates or to significantly improve the reliability of a wireless link. In his seminal paper, Telatar showed that in case of independent and identically distributed Rayleigh-fading, the ergodic capacity of a MIMO channel grows linearly with the minimum out of the number of transmit and receive antennas, respectively [1]. Later, his results were extended by Shin and Lee [2] as well as Kiessling and Speidel [3], [4] to scenarios with spatial correlation at the transmitting and/or the receiving antenna array, in which case the capacity generally might be drastically reduced. However, spatial correlation is not the only detrimental factor that may derogate the capacity of a MIMO channel in practice. In fact, it has been predicted theoretically in [5] and [6] and verified by several different measurement campaigns (see for example [7]) that there might be scenarios in which the MIMO channel matrix has only low rank, even though the transmit and receive signals are only weakly correlated or even totally uncorrelated. This phenomenon is usually referred to as *keyhole effect* and might occur due to various different propagation effects, such as diffraction or waveguiding, for example [6].

The capacity of uncorrelated keyhole channels with additive white Gaussian noise (AWGN) has already been investigated in [2] whereas a lower bound on the ergodic capacity of correlated keyhole channels has been presented in [8]. However, in both cases the authors confined themselves to considering the ergodic channel capacity only while a comprehensive analysis of the statistical properties of the mutual information has, to the best of our knowledge, not been addressed in literature before. In this paper, we perform such an analysis by deriving concise mathematical closed-form expressions for the probability density function (pdf) and cumulative distribution function (cdf) of the mutual information of arbitrarily correlated Rayleigh-fading MIMO channels with keyhole, considering the most general case with arbitrary input covariance matrices and possibly colored additive Gaussian noise. Furthermore, we calculate the corresponding mean value, based on which the *exact* ergodic channel capacity can easily be determined.

The remainder of this paper is structured as follows: In Section II, we introduce our system and channel model. The actual statistical analysis of the mutual information is done in Section III, followed by some numerical results in Section IV. Finally, some concluding remarks are given in Section V.

Notation: Vectors and matrices are denoted by bold lower and upper case letters, respectively. $\mathcal{C}_N^m(\mathbf{x}, \mathbf{R}_{xx})$ represents an m-variate complex Gaussian random vector with mean $\mathbf{x}$ and covariance matrix $\mathbf{R}_{xx}$, $\mathbf{I}_n$ is the identity matrix of size $n \times n$, the superscript $(\cdot)^H$ stands for the conjugate-transpose of a matrix or vector, and $\mathbb{E}[\cdot]$ represents the expectation operator. $\mathbf{A}^{1/2}$ and $\mathbf{A}^{H/2}$ denote matrix roots of matrix $\mathbf{A}$ such that $\mathbf{A}^{1/2}\mathbf{A}^{H/2} = \mathbf{A}$, $X \sim$ means "random variable X is distributed as" while $X \cong Y$ means "random variable X is statistically equivalent to random variable Y". Finally, $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ denote the trace and determinant of a matrix, respectively.

II. System and Channel Model

We consider a frequency-flat Rayleigh-fading MIMO channel with N_{TX} transmit antennas and N_{RX} receive antennas. In the discrete-time equivalent baseband domain, the system can be described during one channel use by

$$\mathbf{r} = \sqrt{\frac{\bar{\gamma}}{N_{TX}}}\,\mathbf{H}\,\mathbf{s} + \mathbf{n}, \qquad (1)$$

where $\mathbf{H} \in \mathbb{C}^{N_{RX} \times N_{TX}}$ denotes the channel matrix, $\mathbf{s} \in \mathbb{C}^{N_{TX}}$ the transmit signal, $\mathbf{r} \in \mathbb{C}^{N_{RX}}$ the received signal, $\mathbf{n} \in \mathbb{C}^{N_{RX}}$ an additive Gaussian noise vector, and $\bar{\gamma}$ the average signal-to-noise ratio (SNR) per receive antenna. The transmit signals $\mathbf{s}$ are assumed to have zero mean and arbitrary autocorrelation matrix $\mathbf{R}_{ss} = \mathbb{E}\left[\mathbf{s}\mathbf{s}^H\right]$, where we request without loss of generality that this matrix is normalized such that $\mathrm{tr}(\mathbf{R}_{ss}) = N_{TX}$. Furthermore, we assume that the additive noise has zero mean as well and that its autocorrelation matrix is given by $\mathbf{R}_{nn} = \mathbb{E}\left[\mathbf{n}\mathbf{n}^H\right]$, which we request to have full rank and to be normalized such that $\mathrm{tr}(\mathbf{R}_{nn}) = N_{RX}$. Please note that the

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requested normalizations of $\mathbf{R}_{ss}$ and $\mathbf{R}_{nn}$ are always feasible by properly adjusting the average SNR $\bar{\gamma}$, which we define as $\bar{\gamma} = P_T/N_0$, where P_T and N_0 denote the total transmit power and the average noise power per receive antenna, respectively. Furthermore, we assume a perfect keyhole channel, i.e., the only way for the radio waves to propagate from the transmitter to the receiver is to pass through a keyhole (e.g., a hallway acting as a single-mode waveguide), which ideally re-radiates all the captured energy. In this case, the channel can be considered as a concatenation of a multiple-input single-output (MISO) channel from the individual transmit antennas to the keyhole and a (statistically independent) single-input multiple-output (SIMO) channel from the keyhole to the various receive antennas [5], i.e., the channel matrix $\mathbf{H}$ can be modeled as

$$\mathbf{H} = \mathbf{x}\,\mathbf{y}^H, \tag{2}$$

where the vectors $\mathbf{x} \in \mathbb{C}^{N_{RX}}$ and $\mathbf{y} \in \mathbb{C}^{N_{TX}}$ denote the aforementioned SIMO and MISO channels, respectively. Considering a spatially correlated Rayleigh-fading scenario, we have $\mathbf{x} \sim \mathcal{C}_N^{N_{RX}}(\mathbf{0}, \mathbf{R}_{RX})$ and $\mathbf{y} \sim \mathcal{C}_N^{N_{TX}}(\mathbf{0}, \mathbf{R}_{TX})$, where $\mathbf{R}_{TX}$ and $\mathbf{R}_{RX}$ denote the correlation matrices at the transmitter and the receiver, respectively. In the following, the channel is always assumed to be perfectly known by the receiver while it is unknown to the transmitter.

III. Mutual Information Analysis

A. Probability Density Function

From [1], [4], it is well-known that for a fixed realization of the channel matrix $\mathbf{H}$, the mutual information (in bits per channel use) between the input signal vector $\mathbf{s}$ and the output signal vector $\mathbf{r}$ of the considered MIMO system is given by

$$I(\mathbf{s}, \mathbf{r}) = \log_2 \det\left(\mathbf{I}_{N_{TX}} + \frac{\bar{\gamma}}{N_{TX}}\,\mathbf{R}_{ss}\,\mathbf{H}^H\,\mathbf{R}_{nn}^{-1}\,\mathbf{H}\right). \tag{3}$$

Exploiting the special structure of $\mathbf{H}$ in case of keyhole channels as introduced in (2), we can reformulate (3) as

$$I(\mathbf{s}, \mathbf{r}) = \log_2 \det\left(\mathbf{I}_{N_{TX}} + \frac{\bar{\gamma}}{N_{TX}}\,\mathbf{R}_{ss}\,\mathbf{y}\,\mathbf{x}^H\,\mathbf{R}_{nn}^{-1}\,\mathbf{x}\,\mathbf{y}^H\right). \tag{4}$$

Since $\mathbf{x}$ and $\mathbf{y}$ are complex Gaussian random vectors with zero mean and covariance matrix $\mathbf{R}_{RX}$ and $\mathbf{R}_{TX}$, respectively, it can easily be shown that

$$I(\mathbf{s}, \mathbf{r}) \cong \log_2 \det\left(\mathbf{I}_{N_{TX}} + \frac{\bar{\gamma}}{N_{TX}}\,\tilde{\mathbf{y}}\,\tilde{\mathbf{x}}^H\,\tilde{\mathbf{x}}\,\tilde{\mathbf{y}}^H\right), \tag{5}$$

with random vectors $\tilde{\mathbf{x}} \sim \mathcal{C}_N^{N_{RX}}(\mathbf{0}, \mathbf{R}_{nn}^{-H/2}\mathbf{R}_{RX}\mathbf{R}_{nn}^{-1/2})$ and $\tilde{\mathbf{y}} \sim \mathcal{C}_N^{N_{TX}}(\mathbf{0}, \mathbf{R}_{ss}^{H/2}\mathbf{R}_{TX}\mathbf{R}_{ss}^{1/2})$. Making use of the determinant identity $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$ for arbitrary matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}\,\mathbf{B}$ is square [9], (5) can be written as

$$I(\mathbf{s}, \mathbf{r}) \cong \log_2\left(1 + \frac{\bar{\gamma}}{N_{TX}}\,||\tilde{\mathbf{x}}||^2\,||\tilde{\mathbf{y}}||^2\right), \tag{6}$$

where $||\cdot||^2$ denotes the squared Euclidean norm of a vector. Hence, the distribution of $I(\mathbf{s}, \mathbf{r})$ is obviously directly connected to the distribution of the random variable $\Phi = ||\tilde{\mathbf{x}}||^2\,||\tilde{\mathbf{y}}||^2$, which is provided by the following lemma:

Lemma 1: The probability density function of the random variable $\Phi = ||\tilde{\mathbf{x}}||^2\,||\tilde{\mathbf{y}}||^2$ in (6) is given by

$$\begin{aligned} p_\Phi(\phi) &= \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{2\,\xi_{i,j}\,\zeta_{k,l}}{\Gamma(j)\,\Gamma(l)\,(\lambda_i\,\sigma_k)^{\frac{j+l}{2}}} \\ &\quad \times \phi^{\frac{j+l}{2}-1}\,K_{j-l}\left(2\sqrt{\frac{\phi}{\lambda_i\,\sigma_k}}\right), \quad \phi \geq 0 \end{aligned} \tag{7}$$

where the coefficients λ_i denote the M distinct m_i-fold non-zero eigenvalues of the matrix $\mathbf{R}_{nn}^{-1}\,\mathbf{R}_{RX}$ and the coefficients σ_k are the N distinct n_k-fold non-zero eigenvalues of the matrix $\mathbf{R}_{ss}\,\mathbf{R}_{TX}$. Furthermore, we have

$$\xi_{i,j} = \frac{(-\lambda_i)^{j-m_i}}{(m_i - j)!}\,\frac{\partial^{m_i - j}}{\partial s^{m_i - j}}\left[\prod_{\substack{v=1\\v\neq i}}^{M} \frac{1}{(1 - s\,\lambda_v)^{m_v}}\right]\Bigg|_{s=\frac{1}{\lambda_i}} \tag{8}$$

$$\zeta_{k,l} = \frac{(-\sigma_k)^{l-n_k}}{(n_k - l)!}\,\frac{\partial^{n_k - l}}{\partial s^{n_k - l}}\left[\prod_{\substack{v=1\\v\neq k}}^{N} \frac{1}{(1 - s\,\sigma_v)^{n_v}}\right]\Bigg|_{s=\frac{1}{\sigma_k}} \tag{9}$$

and $\Gamma(\cdot)$ as well as $K_\nu(\cdot)$ denote the gamma function and the ν-th order modified Bessel function of the second kind, respectively [10].

Proof: For proving Lemma 1, we first of all consider the distribution of $X = ||\tilde{\mathbf{x}}||^2$. Pursuing a similar approach as presented in [11], it can easily be shown that the moment-generating function (mgf) of X is given by

$$M_X(s) = \frac{1}{\det\left(\mathbf{I}_{N_{RX}} - s\,\mathbf{R}_{nn}^{-H/2}\,\mathbf{R}_{RX}\,\mathbf{R}_{nn}^{-1/2}\right)}. \tag{10}$$

In order to simplify this expression, we note that the determinant of a matrix is always given by the product of its eigenvalues and that the non-zero eigenvalues of $\mathbf{A}\,\mathbf{B}$ and their multiplicities are identical to those of $\mathbf{B}\,\mathbf{A}$ (provided that $\mathbf{A}$ and $\mathbf{B}$ have appropriate dimensions such that $\mathbf{A}\,\mathbf{B}$ is square) [9]. Assuming without loss of generality that the matrix $\mathbf{R}_{nn}^{-1}\,\mathbf{R}_{RX}$ has $M \leq N_{RX}$ distinct non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$ with multiplicities $m_1, m_2, \ldots, m_M$, it is then straightforward to rewrite the mgf of X according to (10) as

$$M_X(s) = \prod_{i=1}^{M} \frac{1}{(1 - s\,\lambda_i)^{m_i}}. \tag{11}$$

Expanding this term into partial fractions in order to get rid of the product, we obtain

$$M_X(s) = \sum_{i=1}^{M}\sum_{j=1}^{m_i} \frac{\xi_{i,j}}{(1 - s\,\lambda_i)^j}, \tag{12}$$

where the expansion coefficients $\xi_{i,j}$ can be calculated analytically by means of (8). Based on this result, the desired pdf of X can then be obtained by performing the inverse Laplace transform of $M_X(-s)$, yielding to

$$p_X(x) = \sum_{i=1}^{M}\sum_{j=1}^{m_i} \frac{\xi_{i,j}\,x^{j-1}}{\Gamma(j)\,\lambda_i^j}\,e^{-\frac{x}{\lambda_i}}, \quad x \geq 0. \tag{13}$$

Likewise, the PDF of $Y = ||\tilde{\mathbf{y}}||^2$ can be shown to be given as

$$p_Y(y) = \sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{\zeta_{k,l}\, y^{k-1}}{\Gamma(l)\,\sigma_k^l}\, e^{-\frac{y}{\sigma_k}}, \quad y \geq 0, \tag{14}$$

where we assumed without loss of generality that the matrix $\mathbf{R}_{ss}\,\mathbf{R}_{TX}$ has N distinct non-zero eigenvalues $\sigma_1, \sigma_2, \ldots, \sigma_N$ with multiplicities $n_1, n_2, \ldots, n_N$ and where the coefficients $\zeta_{k,l}$ can be calculated based on (9). Exploiting that X and Y are statistically independent of each other, the desired pdf of $\Phi = ||\tilde{\mathbf{x}}||^2\, ||\tilde{\mathbf{y}}||^2 = X\,Y$ can then be determined as

$$\begin{aligned} p_\Phi(\phi) &= \int_0^\infty p_X(x)\, p_Y\!\left(\frac{\phi}{x}\right) \frac{1}{|x|}\, dx \qquad (15)\\ &= \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{\xi_{i,j}\,\zeta_{k,l}\,\phi^{l-1}}{\Gamma(j)\,\Gamma(l)\,\lambda_i^j\,\sigma_k^l} \\ &\quad \times \int_0^\infty x^{j-l-1}\, e^{-\left(\frac{x}{\lambda_i} + \frac{\phi}{x\,\sigma_k}\right)}\, dx \quad \phi \geq 0. \qquad (16) \end{aligned}$$

Making use of [10] eq. (3.471,9), this integral can be solved in closed-form, yielding to the result provided in (7), what eventually concludes the proof. ■

Based on this result, we now can easily determine the desired pdf of $I(\mathbf{s}, \mathbf{r})$, which is stated by the following theorem:

Theorem 1: The probability density function of the mutual information $I(\mathbf{s}, \mathbf{r})$ between the input signal vector $\mathbf{s}$ and the output signal vector $\mathbf{r}$ is given by

$$\begin{aligned} p_I(R) &= \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{\xi_{i,j}\,\zeta_{k,l}}{\Gamma(j)\,\Gamma(l)} \left(\frac{\left(2^R - 1\right) N_{TX}}{\lambda_i\,\sigma_k\,\bar{\gamma}}\right)^{\frac{j+l}{2}} \\ &\quad \times \frac{2^{R+1}\,\ln(2)}{(2^R - 1)}\, K_{j-l}\left(2\sqrt{\frac{(2^R - 1)\; N_{TX}}{\lambda_i\,\sigma_k\,\bar{\gamma}}}\right). \qquad (17) \end{aligned}$$

Proof: Performing the simple transformation of random variables $I = \log_2\left(1 + \frac{\bar{\gamma}}{N_{TX}}\,\Phi\right)$, we obtain

$$p_I(R) = \ln\, 2 \cdot 2^R\, \frac{N_{TX}}{\bar{\gamma}}\, p_\Phi\left(\left(2^R - 1\right)\, \frac{N_{TX}}{\bar{\gamma}}\right), \tag{18}$$

what exactly corresponds to (17) after substituting the pdf of Φ with the expression given in Lemma 1. ■

B. Cumulative Distribution Function

The cdf of the mutual information generally corresponds to the probability that a certain information rate R cannot be supported by an instantaneous realization of the channel matrix $\mathbf{H}$. Often, this probability is also referred to as the information outage probability of the channel and our main result in this regard is stated by the following theorem:

Theorem 2: The cumulative distribution function of the mutual information $I(\mathbf{s}, \mathbf{r})$ is given by

$$\begin{aligned} F_I(R) &= 1 - \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k}\sum_{v=0}^{l-1} \frac{2\,\xi_{i,j}\,\zeta_{k,l}}{\Gamma(j)\,\Gamma(v+1)} \\ &\quad \times \left(\frac{\Lambda(R)}{\lambda_i\,\sigma_k}\right)^{\frac{j+v}{2}} K_{j-v}\left(2\sqrt{\frac{\Lambda(R)}{\lambda_i\,\sigma_k}}\right), \qquad (19) \end{aligned}$$

where we have introduced for brevity the short-hand notation

$$\Lambda(R) = \left(2^R - 1\right)\, \frac{N_{TX}}{\bar{\gamma}}, \tag{20}$$

and with $K_\nu(\cdot)$ as the ν-th order modified Bessel function of the second kind again.

Proof: Generally, the cdf $F_I(R)$ of $I(\mathbf{s}, \mathbf{r})$ is defined as $F_I(R)$ = Prob $[I(\mathbf{s}, \mathbf{r}) \leq R]$. It can easily be shown that this is equivalent to $F_I(R)$ = Prob $[\Phi \leq \Lambda(R)]$, with $\Lambda(R)$ according to (20). Based on (16), we consequently obtain

$$F_I(R) = \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{\xi_{i,j}\,\zeta_{k,l}\,\phi^{l-1}}{\Gamma(j)\,\Gamma(l)\,\lambda_i^j\,\sigma_k^l} \cdot \mathcal{I}, \tag{21}$$

with the short-hand notation

$$\mathcal{I} = \int_0^{\Lambda(R)}\int_0^\infty \phi^{l-1}\, x^{j-l-1}\, e^{-\left(\frac{x}{\lambda_i} + \frac{\phi}{x\,\sigma_k}\right)}\, dx\, d\phi, \tag{22}$$

which has been introduced for brevity. Changing the order of integration and capitalizing on [10] eq. (3.381,1), we get

$$\mathcal{I} = \int_0^\infty \sigma_k^l\, x^{j-1}\, e^{-\frac{x}{\lambda_i}}\, \gamma\left(l, \frac{\Lambda(R)}{x\,\sigma_k}\right)\, dx, \tag{23}$$

with $\gamma(a, x) = \int_0^x t^{a-1}\, e^{-t}\, dt$ as the lower incomplete gamma function [10]. Making use of the well-known relationship that $\gamma(n, x)$ is given for positive integers n as

$$\gamma(n, x) = \Gamma(n)\left(1 - e^{-x}\sum_{k=0}^{n-1} \frac{x^k}{\Gamma(k+1)}\right) \tag{24}$$

and exploiting the integral relationships provided in [10] eqs. (3.381,4) and (3.471,9), (23) can be solved in closed-form, whereby we finally obtain the result given in Theorem 2. ■

C. Mean Mutual Information

A central information-theoretic measure of great importance, particularly for characterizing ergodic fading channels, is the mean mutual information, which corresponds to the maximum data rate at which (at least in theory) error-free transmission is possible. Maximizing the mean mutual information over the set of all possible input covariance matrices $\mathbf{R}_{ss}$ then yields the ergodic capacity of the channel. The problem of determining the ergodic capacity of keyhole channels has already been addressed in [2], but for uncorrelated channels and white noise only. In the following, we extend and generalize these results by deriving an exact analytical closed-form expression for the mean mutual information of the considered spatially correlated keyhole channels with possibly colored noise and arbitrary input covariance matrices. Based on this result, the ergodic capacity with uninformed transmitter can then easily be obtained by simply setting $\mathbf{R}_{ss} = \mathbf{I}_{N_{TX}}$.

Theorem 3: The mean mutual information between the input signal vector $\mathbf{s}$ and the output signal vector $\mathbf{r}$ of the considered MIMO channel is given by

$$\begin{aligned} \mathbb{E}\left[I(\mathbf{s}, \mathbf{r})\right] &= \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k} \frac{\xi_{i,j}\,\zeta_{k,l}}{\ln(2)\,\Gamma(j)\,\Gamma(l)} \\ &\quad \times G_{4,2}^{1,4}\left[\frac{\bar{\gamma}\,\lambda_i\,\epsilon_r}{N_{TX}}\,\middle|\, \begin{matrix} 1-j, 1-l, 1, 1 \\ 1, 0 \end{matrix}\right], \qquad (25) \end{aligned}$$

where $G^{p,q}_{m,n}[\cdot|\cdot]$ denotes the Meijer-G function [10].

Proof: Generally, $\mathbb{E}\left[I(\mathbf{s},\mathbf{r})\right]$ can be calculated as

$$\mathbb{E}\left[I(\mathbf{s},\mathbf{r})\right] = \int_0^\infty \log_2\left(1+\frac{\bar{\gamma}}{N_{TX}}\,\phi\right) p_\Phi(\phi)\,d\phi. \quad (26)$$

Replacing $p_\Phi(\phi)$ with the expression provided in (7), we get

$$\begin{aligned}\mathbb{E}\left[I(\mathbf{s},\mathbf{r})\right] &= \sum_{i=1}^{M}\sum_{j=1}^{m_i}\sum_{k=1}^{N}\sum_{l=1}^{n_k}\frac{2\,\xi_{i,j}\,\zeta_{k,l}}{\Gamma(j)\,\Gamma(l)\,\ln(2)\,(\lambda_i\,\sigma_k)^{\frac{j+l}{2}}}\\ &\times\int_0^\infty \ln\left(1+\frac{\bar{\gamma}}{N_{TX}}\,\phi\right)\phi^{\frac{j+l}{2}-1}\\ &\times K_{j-l}\left(2\sqrt{\frac{\phi}{\lambda_i\,\sigma_k}}\right)d\phi. \quad (27)\end{aligned}$$

Due to the relatively complex structure of the integrand, the integral in (27) can—to the best of our knowledge—not be solved in closed-form using standard integration techniques or standard tables of integrals. For that reason, we pursue an approach based upon Meijer G-functions in the following [10], which has already been used in [2]. These functions are of very general nature and contain basically all known elementary functions as special cases. Since they are readily available in common mathematical software packages, a numerical evaluation of these Meijer G-functions is immediately feasible.

From [12] eq. (8.4.6,5), we specifically know the identity

$$\ln(1+x) = G^{1,2}_{2,2}\left[x\,\middle|\,\begin{matrix}1, & 1\\ 1, & 0\end{matrix}\right]. \quad (28)$$

Replacing the logarithm in (27) by this equivalent expression, the integral can be written in a form which finally can be solved in closed-form using the result provided in [10] eq. (7.821,3). This way, we obtain the expression given in (25), what eventually concludes the proof. ∎

Even though the exact analytical expression for the mean mutual information according to (25) might be easily evaluated numerically, it is not very intuitive and does not directly reveal its dependence on the spatial correlation properties of the channel and the input signal and noise covariance matrices, respectively. For that purpose, we determine a simple upper bound as well as the corresponding high signal-to-noise ratio (SNR) asymptotics in the following, which can be expressed by means of elementary functions only.

Theorem 4: A simple upper bound I_{bound} on the mean mutual information $I(\mathbf{s},\mathbf{r})$ is given by

$$I_{\text{bound}} = \log_2\left(1+\frac{\bar{\gamma}}{N_{TX}}\,\text{tr}\left(\mathbf{R}_{nn}^{-1}\,\mathbf{R}_{RX}\right)\,\text{tr}\left(\mathbf{R}_{ss}\,\mathbf{R}_{TX}\right)\right). \quad (29)$$

Proof: Exploiting the concavity of the log-function and applying Jensen's inequality to (26), we obtain

$$I(\mathbf{s},\mathbf{r}) \le I_{\text{bound}} = \log_2\left(1+\frac{\bar{\gamma}}{N_{TX}}\,\mathbb{E}[\Phi]\right). \quad (30)$$

Since $\Phi = ||\tilde{\mathbf{x}}||^2\,||\tilde{\mathbf{y}}||^2$, where $||\tilde{\mathbf{x}}||^2$ and $||\tilde{\mathbf{y}}||^2$ are statistically independent of each other, we can say that

$$I_{\text{bound}} = \log_2\left(1+\frac{\bar{\gamma}}{N_{TX}}\,\mathbb{E}\left[||\tilde{\mathbf{x}}||^2\right]\,\mathbb{E}\left[||\tilde{\mathbf{y}}||^2\right]\right). \quad (31)$$

Exploiting the equivalence $\mathbb{E}[||\mathbf{a}||^2 = \text{tr}\left(\mathbb{E}[\mathbf{a}\,\mathbf{a}^H]\right)$ and the known statistical properties of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ then yields to

$$\begin{aligned}I_{\text{bound}} &= \log_2\left(1+\frac{\bar{\gamma}}{N_{TX}}\,\text{tr}\left(\mathbf{R}_{nn}^{-H/2}\,\mathbf{R}_{RX}\,\mathbf{R}_{nn}^{-1/2}\right)\right.\\ &\left.\times\,\text{tr}\left(\mathbf{R}_{ss}^{H/2}\,\mathbf{R}_{TX}\,\mathbf{R}_{ss}^{1/2}\right)\right). \quad (32)\end{aligned}$$

Based on this expression, the final result given in (29) now can easily by obtained by making use of the fact that tr($\mathbf{AB}$) = tr($\mathbf{BA}$) for matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A\,B}$ is square. ∎

Remark 1: For isotropic input signals and additive *white* Gaussian noise with equal noise power N_0 on all receiver branches, i.e., for $\mathbf{R}_{ss} = \mathbf{I}_{N_{TX}}$ as well as $\mathbf{R}_{nn} = \mathbf{I}_{N_{RX}}$, the upper bound according to (29) simplifies to

$$I_{\text{bound}} = \log_2\left(1+N_{RX}\,\bar{\gamma}\right). \quad (33)$$

This means that in this frequently considered case the mean mutual information of a MIMO keyhole channel is upper bounded by the mean mutual information of an additive white Gaussian noise SIMO channel with N_{RX} receive antennas.

For the asymptotics of the mean mutual information in the high SNR regime, we can formulate the following theorem:

Theorem 5: The high SNR asymptotics I_{high} of the mean mutual information $I(\mathbf{s},\mathbf{r})$ are given by

$$I_{\text{high}} = \log_2\left(\frac{\bar{\gamma}}{N_{TX}}\right) + \Xi_{TX} + \Xi_{RX}, \quad (34)$$

with

$$\Xi_{TX} = \sum_{i=1}^{M}\sum_{j=1}^{m_i}\frac{\xi_{i,j}}{\ln 2}\left[\psi(j)+\ln\,\lambda_i\right] \quad (35)$$

$$\Xi_{RX} = \sum_{k=1}^{N}\sum_{l=1}^{n_k}\frac{\zeta_{k,l}}{\ln 2}\left[\psi(l)+\ln\,\sigma_k\right], \quad (36)$$

where $\psi(\cdot)$ denotes Euler's psi-function, which is given for positive integers n as $\psi(n) = -\epsilon + \sum_{k=1}^{n-1}\frac{1}{k}$ with ϵ as the Euler-Mascheroni constant [10].

Proof: It can easily be seen that in the high SNR regime the mean mutual information can be reasonably approximated by

$$I_{\text{high}} = \mathbb{E}\left[\log_2\left(\frac{\bar{\gamma}}{N_{TX}}\,\Phi\right)\right], \quad (37)$$

where the expectation has to be taken with respect to the distribution of Φ. Noting again that $\Phi = ||\tilde{\mathbf{x}}||^2\,||\tilde{\mathbf{y}}||^2$, (37) can be transformed to (34), where $\Xi_{TX} = \mathbb{E}[\log_2(||\tilde{\mathbf{y}}||^2)]$ and $\Xi_{RX} = \mathbb{E}[\log_2(||\tilde{\mathbf{x}}||^2)]$. These expected values can be calculated in closed-form based on (13) and (14) by making use of [10] eq. (4.352,1), whereby we finally get the expressions given in (35) and (36), respectively. Please note that the asymptotical tightness of (34) can easily be checked by showing that $\lim_{\bar{\gamma}\to\infty}|\mathbb{E}[I(\mathbf{s},\mathbf{r})] - I_{\text{high}}| = 0$, what, however, is not explicitly shown here due to space constraints. ∎

At this point, it is also interesting to note that the high SNR asymptotics always represent a lower bound on the exact ergodic capacity since $\log_2(x) \le \log_2(1+x)\;\forall\,x>0$. Furthermore, it can easily be seen based on (34) that keyhole channels provide no spatial multiplexing gain since $\lim_{\bar{\gamma}\to\infty} \frac{I_{\text{high}}}{\log_2 \bar{\gamma}} = 1$, independent of the actual antenna configuration. In fact, this is quite obvious since keyhole channels provide only a single degree of freedom [2]. In addition, it is getting apparent from (34) that spatial correlation affects only the values of Ξ_{TX} and Ξ_{RX} in the high SNR regime and hence leads to a constant offset of the mean mutual information. In this regard, we can formulate the following corollary:

Corollary 1: Assuming isotropic inputs, i.e., $\mathbf{R}_{ss} = \mathbf{I}_{N_{TX}}$, the difference between the mean mutual information in case of uncorrelated and fully correlated transmit signals is given in the high SNR regime by

$$\Delta_{\text{corr,TX}} = \frac{1}{\ln 2}\left[\sum_{k=1}^{N_{TX}-1} \frac{1}{k} - \ln N_{TX}\right]. \quad (38)$$

Proof: In case of uncorrelated transmit signals, all eigenvalues σ_k of $\mathbf{R}_{TX}$ are equal to one, i.e., we have $N = 1$, $n_1 = N_{TX}$, and $\sigma_1 = 1$. Furthermore, it can easily be shown that

$$\zeta_{1,l} = \begin{cases} 1 & \text{for } l = N_{TX} \\ 0 & \text{otherwise} \end{cases} . \quad (39)$$

Hence, Ξ_{TX} is given by $\Xi_{TX,a} = \frac{1}{\ln 2}\left[-\epsilon + \sum_{k=1}^{N_{TX}-1} \frac{1}{k}\right]$ in this case. With full spatial correlation at the transmitter-side, we have only one non-zero eigenvalue with multiplicity one, which is equal to the number of transmit antennas, i.e., $N = 1$, $n_1 = 1$, $\sigma_1 = N_{TX}$, and $\zeta_{1,1} = 1$. Hence, we obtain $\Xi_{TX,b} = \frac{1}{\ln 2}\left[-\epsilon + \ln N_{TX}\right]$. The result provided by (38) then simply corresponds to the difference between $\Xi_{TX,a}$ and $\Xi_{TX,b}$. ■

Similar considerations can be made for spatial correlation at the receiver-side if we assume AWGN, i.e., $\mathbf{R}_{nn} = \mathbf{I}_{N_{RX}}$, where the corresponding expression for $\Delta_{\text{corr,RX}}$ can be obtained from (38) by simply replacing N_{TX} with N_{RX}. Clearly, (38) reveals that the impact of spatial correlation on the capacity in case of keyhole channels is generally relatively small. For $N_{TX} = 2$, for example, the difference between the uncorrelated and the fully correlated case at the transmitter-side is just about 0.4427 bits per channel use, what is far less than it would be in case of non-keyhole MIMO channels [4].

A self-evident conjecture in this regard is that the offset between the fully correlated and the uncorrelated case is increasing if more antennas are used since in absence of spatial correlation the diversity order increases in that case whereas in case of fully correlated channels the diversity order always equals one. This is manifested by the following corollary:

Corollary 2: $\Delta_{\text{corr,TX}}$ according to (38) is a strictly increasing function of the number of transmit antennas and the limiting value for $N_{TX} \to \infty$ is given by

$$\lim_{N_{TX}\to\infty} \Delta_{\text{corr,TX}} = \frac{\epsilon}{\ln 2}. \quad (40)$$

Proof: The strictly increasing nature of $\Delta_{\text{corr,TX}}$ can easily be shown by means of complete induction, what is not explicitly presented here due to space constraints. For obtaining the limiting value in (40), we make use of [10] eq. (8.367,2), what directly leads to the given result. ■

Please note that exactly the same considerations can be done for the receiver-side again, i.e., similarly to (40) we can say that $\lim_{N_{RX}\to\infty} \Delta_{\text{corr,RX}} = \frac{\epsilon}{\ln 2}$.

IV. Numerical Results

In the following, we restrict for simplicity to considering isotropic input signals and AWGN only, i.e., we assume that $\mathbf{R}_{ss} = \mathbf{I}_{N_{TX}}$ and $\mathbf{R}_{nn} = \mathbf{I}_{N_{RX}}$. Furthermore, for investigating the impact of spatial correlation, we assume an exponential correlation model, i.e., the entry in the m-th row and n-th column of the transmit correlation matrix $\mathbf{R}_{TX}$ is given by $[\mathbf{R}_{TX}]_{m,n} = \rho_{TX}^{|m-n|}$ $(0 \le \rho_{TX} \le 1)$ and the corresponding entry of $\mathbf{R}_{RX}$ by $[\mathbf{R}_{RX}]_{m,n} = \rho_{RX}^{|m-n|}$ $(0 \le \rho_{RX} \le 1)$, where ρ_{TX} and ρ_{RX} denote two correlation coefficients that can be used for adjusting the degree of correlation.

Fig. 1 shows the mean mutual information versus the average SNR for various antenna configurations with no correlation at the receiver-side and only moderate correlation at the transmitter-side ($\rho_{TX} = 0.7$). Aside from our analytically calculated values, also the corresponding high SNR asymptotics, the upper bound, as well as results obtained from Monte-Carlo simulations are shown. Obviously, there is a perfect match between calculated and simulated values, what verifies the accuracy of our theoretical analysis. Furthermore, it can be seen that our high SNR asymptotics are even for rather moderate average SNRs quite tight and that the accuracy of the upper bound increases with increasing numbers of antennas.

Some examples for the shapes of the pdfs of the mutual information are depicted in Fig. 2. As before, we compare our analytical results with simulated values and it turns out that there is again a perfect agreement between both of them.

Fig. 3 illustrates the impact of spatial correlation at the transmitter-side on the cdf of the mutual information. As can be seen, with increasing spatial correlation, i.e., with increasing values of ρ_{TX}, the probability that the channel supports only rather low rates is increasing, but at the same time also the probability that it supports relatively high rates is

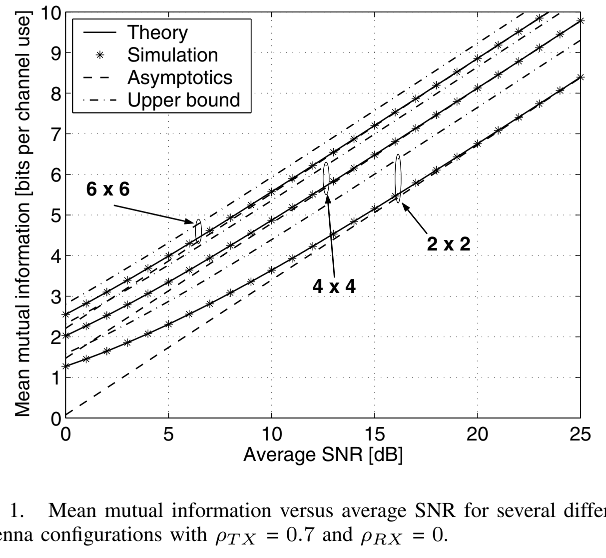

Fig. 1. Mean mutual information versus average SNR for several different antenna configurations with $\rho_{TX} = 0.7$ and $\rho_{RX} = 0$.

getting higher. This can be explained as follows: Since keyhole channels provide no spatial multiplexing gain, the channel matrix $\mathbf{H}$ has always only a single non-zero eigenvalue and consequently every keyhole channel can be transformed to an equivalent SISO channel. The variations of the channel gain of this equivalent SISO channel are generally increasing with increasing spatial correlation, wherefore also the variability of the mutual information increases in that case.

Finally, Fig. 4 shows an example for the information outage probability of correlated keyhole channels versus the average SNR. As can be seen, spatial correlation results in an effective SNR loss in the high SNR regime, i.e., the corresponding curves for different values of ρ_{TX} are simply shifted versions of the one for $\rho_{TX} = 0$. Furthermore, we observe that there is a perfect match between calculated and simulated values again, what proves the accuracy of our analytical results.

V. Conclusion

We have determined concise mathematical closed-form expressions for the pdf, cdf, and expected value of the mutual information of spatially correlated Rayleigh-fading MIMO channels with keyhole, considering the most general case with arbitrarily correlated input signals and potentially colored additive Gaussian noise. Furthermore, a simple upper bound on the mean mutual information as well as the corresponding high SNR asymptotics have been derived, which are more intuitive and easier to evaluate than the exact expression. Numerical results were shown to be in perfect agreement with simulation results, thus verifying the accuracy of our theoretical analysis.

References

[1] İ. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecommun.*, vol. 10, no. 6, pp. 586 – 595, Nov. 1999.

[2] H. Shin and J. H. Lee, "Capacity of multiple-antenna fading channels: spatial fading correlation, double scattering, and keyhole," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2636–2647, Oct. 2003.

[3] M. Kiessling and J. Speidel, "Mutual information of MIMO channels in correlated Rayleigh fading environments - a general solution," in *Proc. IEEE Int. Conf. on Communications*, June 2004.

[4] M. Kiessling, "Unifying analysis of ergodic MIMO capacity in correlated Rayleigh fading environments," *Europ. Trans. Telecommun.*, vol. 16, no. 1, pp. 17 – 35, Jan. 2005.

[5] D. Gesbert, H. Bölcskei, D. A. Gore, and A. J. Paulraj, "Outdoor MIMO wireless channels: models and performance prediction," *IEEE Trans. Commun.*, vol. 50, no. 12, pp. 1926 – 1934, Dec. 2002.

[6] D. Chizhik, G. Foschini, M. Gans, and R. Valenzuela, "Keyholes, correlations, and capacities of multielement transmit and receive antennas," *IEEE Trans. Wireless Commun.*, vol. 1, no. 2, pp. 361 – 368, Apr. 2002.

[7] P. Almers, F. Tufvesson, and A. F. Molisch, "Keyhole effect in MIMO wireless channels: measurements and theory," *IEEE Trans. Wireless Commun.*, vol. 5, no. 12, pp. 3596 – 3604, Dec. 2006.

[8] X. W. Cui and Z. M. Feng, "Lower capacity bound for MIMO correlated fading channels with keyhole," *IEEE Commun. Lett.*, vol. 8, no. 8, pp. 500 – 502, Aug. 2004.

[9] R. A. Horn and C. R. Johnson, *Matrix analysis*, 1st ed. Cambridge University Press, 1985.

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 6th ed. New York: Academic Press, 2000.

[11] J. Luo, J. Zeidler, and S. McLaughlin, "Performance analysis of compact antenna arrays with MRC in correlated Nakagami fading channels," *IEEE Trans. Veh. Technol.*, vol. 50, no. 1, pp. 267 – 277, Jan. 2001.

[12] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series, Volume 3: More Special Functions*. Gordon and Breach Science Publishers, 1990.

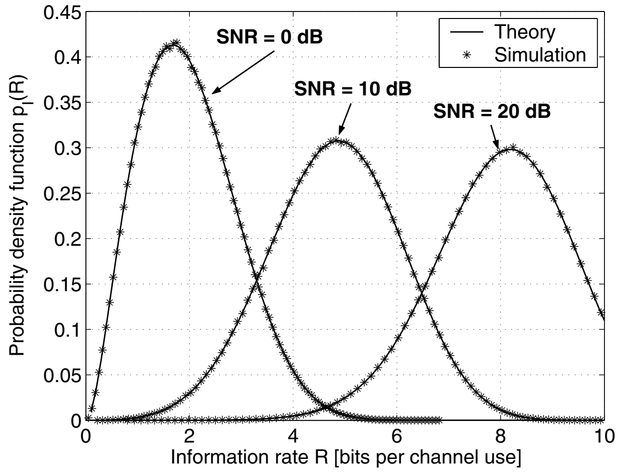

Fig. 2. Probability density function of the mutual information for $N_{TX} = N_{RX} = 4$, $\rho_{TX} = 0.7$, $\rho_{RX} = 0.5$, and various average SNRs $\bar{\gamma}$.

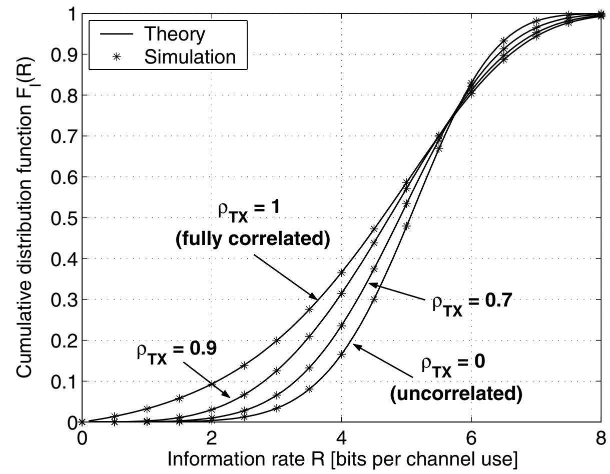

Fig. 3. Cumulative distribution function of the mutual information for $N_{TX} = N_{RX} = 4$, $\rho_{RX} = 0$, $\bar{\gamma} = 10\,\text{dB}$, and several different values of ρ_{TX}.

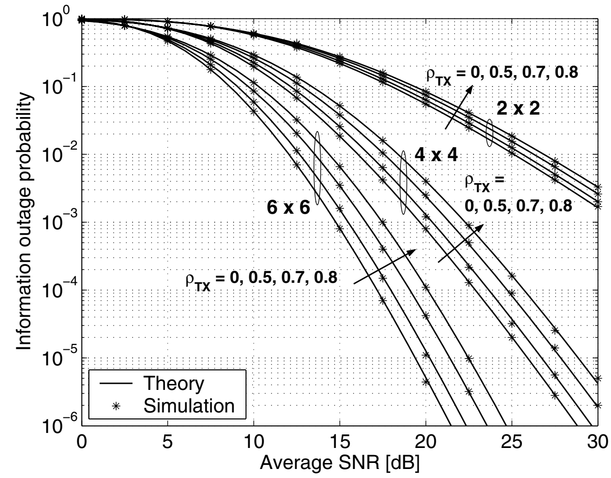

Fig. 4. Information outage probability for $\rho_{RX} = 0.5$, $R = 4$ bits per channel use, and several different antenna configurations as well as transmit correlation coefficients ρ_{TX}.