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Ergodic Capacity and Information Outage Probability of MIMO Nakagami-m Keyhole Channels with General Branch Parameters

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***Abstract*— We derive exact analytical closed-form expressions for the ergodic capacity and information outage probability of multiple-input multiple-output (MIMO) keyhole channels in Nakagami-m fading environments. In this regard, we consider the most general case with channel coefficients having not necessarily identical fading parameters and average power gains, respectively. The ergodic capacity is given as a finite sum of weighted Meijer G-functions, which might be easily evaluated numerically. Additionally, we provide somewhat simpler upper and lower bounds, which can be expressed by means of elementary functions only and which are proven to be asymptotically tight for high signal-to-noise ratios. Numerical results are shown to be in perfect agreement with results obtained from Monte-Carlo simulations, thus verifying the accuracy of our theoretical analysis.**

I. Introduction

Wireless communication systems employing multiple antennas at both the transmitter- and the receiver-side of a radio link are known to offer a wide variety of benefits and particularly significant performance advantages over conventional single-input single-output systems. In his seminal paper [1], Telatar showed that in case of independent and identically distributed (IID) Rayleigh-fading, the ergodic capacity of multiple-input multiple-output (MIMO) channels scales linearly with the minimum out of the number of transmit and receive antennas, respectively. However, in real-world propagation scenarios the actual capacity gains are often much smaller than reported in [1] due to the detrimental effects of spatial fading correlation, for instance, whose impact on the ergodic capacity of Rayleigh-fading MIMO channels has been intensively studied in literature over the past few years, see for example [2]–[4].

Another effect that might lead to a capacity reduction of MIMO channels in practice are possible degenerate channel phenomena, frequently also referred to as *keyholes* or *pinholes* [5], [6]. Such keyhole channels generally characterize rank-deficient MIMO channels, which may have sufficient scattering around the transmitter and receiver to obtain uncorrelated or at least only weakly correlated signals, but due to other propagation effects, such as diffraction or waveguiding, the channel matrix might nevertheless exhibit only low rank. Therefore, it is quite obvious that keyholes might have a considerable impact on the performance of any MIMO system. The existence of this effect has been predicted theoretically in [5] and [6], for instance, and it has been verified by several different measurement campaigns, see for example [7] and [8].

In this paper, we derive exact analytical closed-form expressions for the ergodic capacity and information outage probability of MIMO Nakagami-m keyhole channels, where we consider the most general case with channel coefficients having not necessarily identical fading parameters and average power gains, respectively. In practice, fading and power imbalances might occur in case of largely-spaced antennas, for example, or in case of distributed MIMO systems, where multiple—not necessarily co-located—single-antenna users cooperate in such a way that a virtual MIMO system is established.

As an important information-theoretic measure, the ergodic capacity generally reflects the maximum information rate for which—at least in theory—error-free transmission is possible, provided that the length of one coding block is sufficiently large to cover all possible fading states. However, for that reason it is basically only suitable for characterizing ergodic fading channels whereas it is usually more expedient to consider the information outage probability in case of non-ergodic channels instead. This measure represents the probability that a certain information rate cannot be supported by an instantaneous realization of the channel and for IID Rayleigh-fading MIMO channels, the problem of finding this probability has been addressed in [9] and [10], for example. A closed-form expression for the ergodic capacity of Rayleigh-fading MIMO channels with keyhole has already been reported in [2], which represents a special case of the results that we derive herein. The problem of finding analytical closed-form expressions for the information outage probability of keyhole channels has, however, to the best of our knowledge not been addressed in literature before, what we therefore will do herein.

The remainder of this paper is structured as follows: In Section II, we outline our system and channel model. The actual ergodic capacity analysis is done in Section III whereas the corresponding information outage probability is derived in Section IV. Finally, numerical results are presented in Section V, followed by some concluding remarks in Section VI.

II. System and Channel Model

We consider a frequency-flat MIMO communication system with N_{TX} transmit antennas and N_{RX} receive antennas. In the discrete-time equivalent baseband domain, the channel can be modeled by the matrix $\mathbf{H} \in \mathbb{C}^{N_{RX} \times N_{TX}}$ and the input-output relationship of our system during one channel use can be expressed as $\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}$, where $\mathbf{y} \in \mathbb{C}^{N_{RX}}$ denotes the received signal vector, $\mathbf{s} \in \mathbb{C}^{N_{TX}}$ the transmitted signal vector, and $\mathbf{n} \in \mathbb{C}^{N_{RX}}$ an additive white Gaussian noise (AWGN) vector, whose elements are IID complex Gaussian distributed with zero mean and variance σ_n^2. The transmit signal vector $\mathbf{s}$ is assumed to be composed of N_{TX} statistically independent equal power components, each being circularly symmetric complex Gaussian distributed with zero mean, subject to the average transmit power constraint

$$\mathbb{E}[\mathbf{s}^H \mathbf{s}] = P_T, \tag{1}$$

where $\mathbf{s}^H$ denotes the conjugate-transpose of $\mathbf{s}$ and $\mathbb{E}[\cdot]$ the expectation operator. Furthermore, we assume a perfect keyhole channel, i.e., the only way for the radio waves to propagate from the transmitter to the receiver is to pass through a keyhole (e.g., a hallway acting as a single mode waveguide), which reradiates all the captured energy. In this case, the channel can be considered as a concatenation of a multiple-input single-output (MISO) channel from the various transmit antennas to the keyhole and a (statistically independent) single-input multiple-output (SIMO) channel from the keyhole to the various receive antennas [5], i.e., $\mathbf{H} = \mathbf{h}_1 \mathbf{h}_2^H$, where $\mathbf{h}_1 \in \mathbb{C}^{N_{RX}}$ as well as $\mathbf{h}_2 \in \mathbb{C}^{N_{TX}}$ denote the aforementioned SIMO and MISO channels, respectively. In this regard, we assume that the individual elements of $\mathbf{h}_1$ and $\mathbf{h}_2$ are statistically independent of each other, what is approximately fulfilled if the distance between the corresponding antenna elements is sufficiently large. The phases of all elements of $\mathbf{h}_1$ and $\mathbf{h}_2$ are supposed to be uniformly distributed in $[0; 2\pi)$ whereas the corresponding magnitudes are modeled as Nakagami-m variates with general fading parameters and average power gains. The fading parameter and average power gain of the magnitude of the i-th component of $\mathbf{h}_1$ will be denoted by $m_{1,i} \geq \frac{1}{2}$ and $\Omega_{1,i}$ $(1 \leq i \leq N_{RX})$ in the following whereas the corresponding parameters of the j-th component of $\mathbf{h}_2$ will be designated as $m_{2,j} \geq \frac{1}{2}$ and $\Omega_{2,j}$ $(1 \leq j \leq N_{TX})$, respectively. Without loss of generality, we assume that the channel is normalized such that

$$\mathbb{E}[||\mathbf{H}||_F^2] = \mathbb{E}[||\mathbf{h}_1||^2]\, \mathbb{E}[||\mathbf{h}_2||^2] = N_{TX}\, N_{RX} \tag{2}$$

with $||\cdot||_F^2$ as the squared Frobenius norm of a matrix. Finally, we assume that the channel is always perfectly known by the receiver whereas it is unknown to the transmitter and we define the average SNR per receive antenna as $\bar{\gamma} = P_T/\sigma_n^2$.

III. Ergodic Capacity Analysis

A. Exact Analysis

It is well-known that for a fixed realization of the channel matrix $\mathbf{H}$, the mutual information between the input signal vector $\mathbf{s}$ and the output signal vector $\mathbf{y}$ is given by [1], [2], [9]

$$I(\mathbf{s};\mathbf{y}) = \log_2 \det \left(\mathbf{I}_{N_{RX}} + \frac{\bar{\gamma}}{N_{TX}} \mathbf{H}\mathbf{H}^H \right), \tag{3}$$

where $\det(\cdot)$ denotes the determinant of a matrix and $\mathbf{I}_n$ the identity matrix of dimension n. Exploiting the special structure of keyhole channels given by $\mathbf{H} = \mathbf{h}_1 \mathbf{h}_2^H$ as elucidated before and further making use of the determinant identity $\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{BA})$ for arbitrary $m \times n$ and $n \times m$ matrices $\mathbf{A}$ and $\mathbf{B}$, respectively, it can easily be shown that (3) is equivalent to

$$I(\mathbf{s};\mathbf{y}) = \log_2 \left(1 + \frac{\bar{\gamma}}{N_{TX}}\, \Xi \right), \tag{4}$$

where $\Xi = ||\mathbf{h}_1||^2\, ||\mathbf{h}_2||^2 = ||\mathbf{H}||_F^2$. Clearly, the distribution of the mutual information $I(\mathbf{s};\mathbf{y})$ is directly connected to the distribution of Ξ, which is provided by the following lemma:

Lemma 1: For the considered channel model, the probability density function (pdf) of $\Xi = ||\mathbf{H}||_F^2$ is given by

$$\begin{aligned} p_\Xi(\xi) \;&=\; \sum_{\nu=1}^{P_1} \sum_{n=1}^{\rho_{1,\nu}} \sum_{\eta=1}^{P_2} \sum_{l=1}^{\rho_{2,\eta}} \frac{2\, \theta_{1,\nu,n}\, \theta_{2,\eta,l}\, \xi^{\frac{n+l}{2}-1}}{\Gamma(n)\,\Gamma(l)\, \left(\phi_{1,\nu}\, \phi_{2,\eta}\right)^{\frac{n+l}{2}}} \\ &\quad \times K_{n-l}\left(2 \sqrt{\frac{\xi}{\phi_{1,\nu}\, \phi_{2,\eta}}} \right), \quad \xi \geq 0 \end{aligned} \tag{5}$$

where $K_\nu(\cdot)$ is the ν-th order modified Bessel function of the second kind, $\Gamma(\cdot)$ the well-known gamma-function [11], and the coefficients $\phi_{1,\nu}$ $(\nu = 1, \ldots, P_1)$ and $\phi_{2,\eta}$ $(\eta = 1, \ldots, P_2)$ denote the $P_1 \leq N_{RX}$ and $P_2 \leq N_{TX}$ distinctive non-zero values of the quotients $\Omega_{1,i}/m_{1,i}$ and $\Omega_{2,i}/m_{2,i}$, respectively. Furthermore, $\rho_{1,\nu}$ and $\rho_{2,\eta}$ represent the sums of all fading levels $m_{1,l}$ and $m_{2,l}$ corresponding to a particular $\phi_{1,\nu}$ or $\phi_{2,\eta}$, i.e., they can be calculated as (with $i \in \{1;2\}$)

$$\rho_{i,k} = \sum_{m_{i,l} \in \mathbb{M}_{i,k}} m_{i,l}, \quad \text{where } \mathbb{M}_{i,k} = \left\{ m_{i,l} \,\middle|\, \frac{\Omega_{i,l}}{m_{i,l}} = \phi_{i,k} \right\}. \tag{6}$$

Finally, the coefficients $\theta_{i,\nu,n}$ $(i \in \{1;2\})$ are given by

$$\begin{aligned} \theta_{i,\nu,n} \;&=\; \frac{1}{(\rho_{i,\nu}-n)!\, \phi_{i,\nu}^{\rho_{i,\nu}-n}} \\ &\quad \times \frac{\partial^{\rho_{i,\nu}-n}}{\partial s^{\rho_{i,\nu}-n}} \left[\prod_{\substack{l=1 \\ l \neq \nu}}^{P_i} \frac{1}{(1 + s\, \phi_{i,l})^{\rho_{i,l}}} \right] \Bigg|_{s = \frac{-1}{\phi_{i,\nu}}} . \end{aligned} \tag{7}$$

Proof: Since the magnitudes of all elements of $\mathbf{h}_1$ and $\mathbf{h}_2$ are assumed to be statistically independent Nakagami-m variates, it can easily be shown that the moment-generating functions (mgf) of $\Psi_1 = ||\mathbf{h}_1||^2$ and $\Psi_2 = ||\mathbf{h}_2||^2$ are given by

$$M_{\Psi_i}(s) = \prod_{l=1}^{\Lambda_i} \left[\frac{1}{1 - s\, \frac{\Omega_{i,l}}{m_{i,l}}} \right]^{m_{i,l}}, \quad (i \in \{1;2\}) \tag{8}$$

where $\Lambda_1 = N_{RX}$ and $\Lambda_2 = N_{TX}$. Without loss of generality, we assume that there are $P_1 \leq N_{RX}$ distinct non-zero values of the quotients $\Omega_{1,i}/m_{1,i}$ and $P_2 \leq N_{TX}$ distinct non-zero values of the quotients $\Omega_{2,i}/m_{2,i}$, which will be denoted by $\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,P_1}$ and $\phi_{2,1}, \phi_{2,2}, \ldots, \phi_{2,P_2}$ in the following. Using the definition of the parameter $\rho_{i,k}$ according to (6), it is hence straightforward to rewrite (8) as

$$M_{\Psi_i}(s) = \prod_{\nu=1}^{P_i} \frac{1}{(1 - s\, \phi_{i,\nu})^{\rho_{i,\nu}}} \quad (i \in \{1;2\}). \tag{9}$$

Hereafter, we always assume that all coefficients $\rho_{i,k}$ according to (6) are positive integers. This holds, for example, if all fading levels $m_{i,l}$ are positive integers, but also for more general cases. Please note that in principle this assumption does not represent a strong limitation since channels for which not all parameters $\rho_{i,k}$ are natural numbers can usually be accurately approximated by channels for which this assumption is fulfilled. If the requested condition is satisfied, we can make use of partial fraction expansion and transform (9) to

$$M_{\Psi_i}(s) = \sum_{\nu=1}^{P_i} \sum_{n=1}^{\rho_{i,\nu}} \frac{\theta_{i,\nu,n}}{(1 - s\,\phi_{i,\nu})^n}, \quad (i \in \{1;2\}) \tag{10}$$

where the expansion coefficients $\theta_{i,\nu,n}$ can be calculated analytically according to (7). The pdf of Ψ_i ($i \in \{1;2\}$) can then be obtained by simply performing the inverse Laplace transform of $M_{\Psi_i}(-s)$, yielding to

$$p_{\Psi_i}(\psi_i) = \sum_{\nu=1}^{P_i} \sum_{n=1}^{\rho_{i,\nu}} \frac{\theta_{i,\nu,n}\,\psi_i^{n-1}}{\Gamma(n)\,\phi_{i,\nu}^n}\, e^{-\frac{\psi_i}{\phi_{i,\nu}}}, \quad \psi_i \geq 0. \tag{11}$$

Exploiting the statistical independence of Ψ_1 and Ψ_2, the pdf of the product $\Xi = \Psi_1\,\Psi_2$ consequently can be calculated as

$$\begin{aligned} p_\Xi(\xi) &= \int_{-\infty}^{\infty} p_{\Psi_1}(\psi_1)\, p_{\Psi_2}\left(\frac{\xi}{\psi_1}\right) \frac{1}{|\psi_1|}\, d\psi_1 && (12)\\ &= \sum_{\nu=1}^{P_1} \sum_{n=1}^{\rho_{1,\nu}} \sum_{\eta=1}^{P_2} \sum_{l=1}^{\rho_{2,\eta}} \frac{\theta_{1,\nu,n}\,\theta_{2,\eta,l}\,\xi^{l-1}}{\Gamma(n)\,\Gamma(l)\,\phi_{1,\nu}^n\,\phi_{2,\eta}^l} \\ &\quad \times \int_0^\infty \psi_1^{n-l-1}\, e^{-\left(\frac{\psi_1}{\phi_{1,\nu}} + \frac{\xi}{\psi_1\,\phi_{2,\eta}}\right)} d\psi_1. && (13)\end{aligned}$$

Making use of [11] eq. (3.471,9) for solving the integral in (13), we finally get the desired pdf given in Lemma 1. ■

Please note that based on this pdf, it is basically straightforward to obtain a closed-form expression for the pdf of the mutual information $I(\mathbf{s};\mathbf{y})$ by performing a simple transformation of random variables according to (4). Herein, however, we are only interested in the ergodic capacity, for which we can formulate the following theorem:

Theorem 1: The ergodic capacity of Nakagami-m fading MIMO channels with keyhole and general—not necessarily identical—fading levels and average power gains can be calculated as (in bits per channel use)

$$\begin{aligned} C_{\text{erg}} &= \sum_{\nu=1}^{P_1} \sum_{n=1}^{\rho_{1,\nu}} \sum_{\eta=1}^{P_2} \sum_{l=1}^{\rho_{2,\eta}} \frac{\theta_{1,\nu,n}\,\theta_{2,\eta,l}}{\ln 2\,\Gamma(n)\,\Gamma(l)} \\ &\quad \times G_{2,4}^{4,1}\left[\frac{N_{TX}}{\bar{\gamma}\,\phi_{1,\nu}\,\phi_{2,\eta}} \,\middle|\, \begin{matrix} 0,1 \\ n,l,0,0 \end{matrix}\right], && (14)\end{aligned}$$

where $G_{m,n}^{p,q}[\cdot|\cdot]$ denotes the Meijer G-function [11].

Proof: Generally, the ergodic channel capacity corresponds to the mean mutual information maximized over the set of all possible input covariance matrices $\mathbf{R}_{ss} = \mathbb{E}[\mathbf{s}\,\mathbf{s}^H]$. In case that the transmitter does not have any channel knowledge, it is well-known that the capacity-achieving transmit strategy is to perform equal power allocation among the various transmit antennas, what we already assumed in Section II. Hence, the ergodic capacity can simply be calculated as $C_{\text{erg}} = \mathbb{E}[I(\mathbf{s};\mathbf{y})]$ in our case, where the expectation has to be taken with respect to the distribution of the channel matrix $\mathbf{H}$. Capitalizing on (4), we consequently get

$$C_{\text{erg}} = \int_0^\infty \log_2\left(1 + \frac{\bar{\gamma}}{N_{TX}}\,\xi\right) p_\Xi(\xi)\, d\xi. \tag{15}$$

By replacing $p_\Xi(\xi)$ with the expression according to (5), it can be seen that the integrand in (15) corresponds to the product of a logarithm, a modified Bessel function of the second kind, and a power function. To the best of our knowledge, this integral cannot be solved analytically in closed-form using standard tables of integrals or standard integration methods. Therefore, we revert to an approach using Meijer G-functions in the following, which has already been used in [2]. Meijer G-functions are of very general nature and contain basically all known elementary functions as special cases [11], [12]. Since they are readily available in common mathematical software packages, a numerical evaluation of these functions is immediately feasible. From [12] eq. (8.4.6,5), we specifically know the functional identity

$$\ln(1+x) = G_{2,2}^{1,2}\left[x \,\middle|\, \begin{matrix} 1, & 1 \\ 1, & 0 \end{matrix}\right]. \tag{16}$$

With this equivalent representation of the logarithm, (15) together with (5) can be written as

$$\begin{aligned} C_{\text{erg}} &= \sum_{\nu=1}^{P_1} \sum_{n=1}^{\rho_{1,\nu}} \sum_{\eta=1}^{P_2} \sum_{l=1}^{\rho_{2,\eta}} \frac{2\,\theta_{1,\nu,n}\,\theta_{2,\eta,l}}{\ln(2)\,\Gamma(n)\,\Gamma(l)\,(\phi_{1,\nu}\,\phi_{2,\eta})^{\frac{n+l}{2}}} \\ &\quad \times \int_0^\infty \xi^{\frac{n+l}{2}-1}\, G_{2,2}^{1,2}\left[\frac{\bar{\gamma}}{N_{TX}}\,\xi \,\middle|\, \begin{matrix} 1, & 1 \\ 1, & 0 \end{matrix}\right] \\ &\quad \times K_{n-l}\left(2\sqrt{\frac{\xi}{\phi_{1,\nu}\,\phi_{2,\eta}}}\right) d\xi. && (17)\end{aligned}$$

Performing the substitution $x = \frac{\xi}{\phi_{1,\nu}\,\phi_{2,\eta}}$ and exploiting [11] eq. (7.821,3), (17) can be solved in closed-form and after reformulating the corresponding result in a slightly more convenient way by making use of [11] eq. (9.31,1), we finally obtain the expression provided by Theorem 1. ■

Even though the exact ergodic capacity expression according to (14) can be easily evaluated numerically, it might be desirable for some purposes to have simpler (approximate) formulas containing elementary functions only, which are more intuitive than (14) and which reveal the dependence of the ergodic capacity on the various channel parameters in a better way. For that reason, we derive an upper as well as a lower bound in the following, which both can be shown to be asymptotically tight for high signal-to-noise ratios.

B. Upper and Lower Bound and High SNR Asymptotics

In a first step, we determine a lower bound on the ergodic capacity, which is given by the following theorem.

Theorem 2: A lower bound on the ergodic channel capacity C_{erg} according to (14) is given by

$$C_{\text{low}} = \log_2\left(1 + \frac{\bar{\gamma}}{N_{TX}}\, e^{\zeta_1+\zeta_2}\right), \tag{18}$$

with the short-hand notation

$$\zeta_i = \sum_{\nu=1}^{P_i} \sum_{n=1}^{\rho_{i,\nu}} \theta_{i,\nu,n} \left[\ln(\phi_{i,\nu}) - \epsilon + \sum_{k=1}^{n-1} \frac{1}{k} \right], \quad (i \in \{1;2\}) \tag{19}$$

where ϵ denotes the Euler-Mascheroni constant [11].

Proof: Basically, the derivation of the lower bound given in Theorem 2 can be done analogously to the derivation of a lower bound on the ergodic capacity of spatially correlated Rayleigh fading channels with keyhole presented in [13]. First, we note that C_{erg} according to (15) might be written as $C_{\text{erg}} = \mathbb{E}\left[\log_2\left(1 + \frac{\bar{\gamma}}{N_{TX}} \exp(\ln \Xi)\right)\right]$. Making use of the known relationship that $\Xi = \Psi_1 \Psi_2$, where Ψ_1 and Ψ_2 are statistically independent of each other and distributed according to (11), and further exploiting that $\log_2(1 + a\, e^x)$ is for any constant $a > 0$ a convex function in x, we obtain by applying Jensen's inequality to the above expression

$$C_{\text{erg}} \geq C_{\text{low}} = \log_2 \left(1 + \frac{\bar{\gamma}}{N_{TX}}\, e^{\mathbb{E}_{\Psi_1}[\ln\, \Psi_1] + \mathbb{E}_{\Psi_2}[\ln\, \Psi_2]} \right). \tag{20}$$

In this regard, the expectation of the logarithm of Ψ_i ($i \in \{1;2\}$) generally can be calculated based on (11) as

$$\mathbb{E}_{\Psi_i}[\ln\, \Psi_i] = \sum_{\nu=1}^{P_i} \sum_{n=1}^{\rho_{i,\nu}} \frac{\theta_{i,\nu,n}}{\Gamma(n)\, \phi_{i,\nu}^n} \int_0^\infty \ln \psi_i\, \psi_i^{n-1}\, e^{-\frac{\psi_i}{\phi_{i,\nu}}}\, d\psi_i, \tag{21}$$

which can be solved analytically in closed-form by capitalizing on [11] eq. (4.352,1). With [11] eq. (8.365,4), we then obtain after some basic mathematical manipulations eventually the final expression according to (18). ■

Theorem 3: An upper bound on the ergodic channel capacity C_{erg} according to (14) is given by

$$C_{\text{up}} = \log_2 \left(\frac{\bar{\gamma}}{N_{TX}} \right) + \frac{1}{\ln 2} \left[\zeta_1 + \zeta_2 + \sqrt{\frac{N_{TX}}{\bar{\gamma}}}\, \chi_1\, \chi_2 \right], \tag{22}$$

where we have introduced for brevity the short-hand notation

$$\chi_i = \sum_{\nu=1}^{P_i} \sum_{n=1}^{\rho_{i,\nu}} \frac{\theta_{i,\nu,n}\, \Gamma\left(n - \frac{1}{2}\right)}{\Gamma(n)\, \sqrt{\phi_{i,\nu}}}, \quad i \in \{1;2\} \tag{23}$$

and with ζ_1 and ζ_2 according to (19).

Proof: First of all, we show that $\ln(1 + x) \leq \ln x + x^{-1/2}\ \forall\ x \geq 0$. For that purpose, we define the function $g(x) = \ln x + x^{-1/2} - \ln(1 + x)$. It can readily be checked that $g(x) \to \infty$ as $x \to 0$ and it is quite obvious that $g(x) \to 0$ as $x \to \infty$. Furthermore, we have $\frac{\partial}{\partial x} g(x) = \frac{1}{2} \frac{2\sqrt{x} - (1+x)}{x^{3/2}\,(1+x)}$, where the denominator is clearly always larger than zero for all $x > 0$ whereas the numerator can easily be shown to be non-negative in this case. Hence, $g(x)$ is a non-increasing function of x and due to the above limits, it essentially must be always larger than zero. This observation directly implies that the aforementioned inequality is fulfilled. Using this inequality in (15), it is hence straightforward to upper-bound C_{erg} by

$$C_{\text{up}} = \frac{1}{\ln 2} \int_0^\infty \left[\ln \frac{\xi\, \bar{\gamma}}{N_{TX}} + \sqrt{\frac{N_{TX}}{\bar{\gamma}\, \xi}} \right] p_\Xi(\xi)\, d\xi. \tag{24}$$

Exploiting that $\Xi = \Psi_1 \Psi_2$, where Ψ_1 and Ψ_2 are statistically independent random variables with pdf according to (11), this expression can be rewritten as

$$\begin{aligned} C_{\text{up}} &= \log_2 \left(\frac{\bar{\gamma}}{N_{TX}} \right) + \int_0^\infty \log_2(\psi_1)\, p_{\Psi_1}(\psi_1)\, d\psi_1 \\ &\quad + \int_0^\infty \log_2(\psi_2)\, p_{\Psi_2}(\psi_2)\, d\psi_2 + \frac{1}{\ln 2} \sqrt{\frac{N_{TX}}{\bar{\gamma}}} \\ &\quad \times \int_0^\infty \frac{p_{\Psi_1}(\psi_1)}{\sqrt{\psi_1}}\, d\psi_1 \int_0^\infty \frac{p_{\Psi_2}(\psi_2)}{\sqrt{\psi_2}}\, d\psi_2. \end{aligned} \tag{25}$$

The first two integrals in (25) basically have already been calculated before and correspond to $\frac{\zeta_1}{\ln 2}$ and $\frac{\zeta_2}{\ln 2}$, respectively. Besides, we obtain for $\mathcal{J}_i = \int_0^\infty \frac{1}{\sqrt{\psi_i}}\, p_{\Psi_i}(\psi_i)\, d\psi_i$ ($i \in \{1;2\}$) by inserting the pdf according to (11)

$$\mathcal{J}_i = \sum_{\nu=1}^{P_i} \sum_{n=1}^{\rho_{i,\nu}} \frac{\theta_{i,\nu,n}}{\Gamma(n)\, \phi_{i,\nu}^n} \int_0^\infty \psi_i^{n-\frac{3}{2}}\, e^{-\frac{\psi_i}{\phi_{i,\nu}}}\, d\psi_i, \tag{26}$$

which can be solved analytically in closed-form using [11] eq. (3.381,4). Putting everything together then finally yields the expression given in (22). ■

Corollary 1: The upper bound C_{up} and the lower bound C_{low} according to (22) and (18), respectively, are both asymptotically tight for $\bar{\gamma} \to \infty$ and the actual high SNR asymptotics are given by

$$C_{\text{high}} = \log_2 \left(\frac{\bar{\gamma}}{N_{TX}} \right) + \frac{1}{\ln 2} \left[\zeta_1 + \zeta_2 \right], \tag{27}$$

with ζ_1 and ζ_2 according to (19).

Proof: From (18), it can easily be seen that $C_{\text{low}} \to C_{\text{high}}$ as $\bar{\gamma} \to \infty$. Furthermore, since χ_1 and χ_2 in (22) are constants and particularly independent of $\bar{\gamma}$, it is quite clear that this is also fulfilled for the upper bound C_{up}. Consequently, since we generally have $C_{\text{low}} \leq C_{\text{erg}} \leq C_{\text{up}}$, this necessarily holds for the exact ergodic capacity C_{erg} as well. ■

An even simpler upper bound on C_{erg} than the one provided by Theorem 3, which, however, is generally not asymptotically tight for high SNRs, is given by the following corollary:

Corollary 2: A simple upper bound C_{bound} on the ergodic channel capacity according to (14) is given by

$$C_{\text{bound}} = \log_2 \left(1 + \bar{\gamma}\, N_{RX} \right). \tag{28}$$

Proof: Exploiting the concavity of the logarithm and making use of Jensen's inequality in conjunction with the requested normalization of the channel according to (2), we obtain the expression given in (28). ■

IV. Information Outage Probability

In case that the channel is non-ergodic, i.e., if the realizations of the channel matrix $\mathbf{H}$ are randomly drawn but remain fixed over the transmission of one codeword, a capacity in the Shannon sense does not exist since there is always a non-zero probability that a given transmission rate R cannot be supported by an instantaneous realization of the channel matrix $\mathbf{H}$. This probability is usually referred to as the (information) outage probability of the channel and the corresponding result for this measure is provided by the following theorem:

Theorem 4: The information outage probability of Nakagami-m fading MIMO channels with keyhole and general branch parameters is given by

$$P_{\text{out}} = 1 - \sum_{\nu=1}^{P_1}\sum_{n=1}^{\rho_{1,\nu}}\sum_{\eta=1}^{P_2}\sum_{l=1}^{\rho_{2,\eta}}\sum_{k=0}^{l-1} \frac{2\,\theta_{1,\nu,n}\,\theta_{2,\eta,l}}{\Gamma(n)\,\Gamma(k+1)} \times \left(\frac{\Upsilon(R)}{\phi_{1,\nu}\,\phi_{2,\eta}}\right)^{\frac{n+k}{2}} K_{n-k}\left(2\sqrt{\frac{\Upsilon(R)}{\phi_{1,\nu}\,\phi_{2,\eta}}}\right) \quad (29)$$

with the short-hand notation

$$\Upsilon(R) = \left(2^R - 1\right)\cdot\frac{N_{TX}}{\bar{\gamma}}. \quad (30)$$

Proof: Generally, the information outage probability is defined as P_{out} = Prob$[I(\mathbf{s};\mathbf{y}) \leq R]$. It can easily be seen from (4) that this is equivalent to P_{out} = Prob$[\Xi \leq \Upsilon(R)]$ = $\int_0^{\Upsilon(R)} p_\Xi(\xi)\,d\xi$, with $\Xi = ||\mathbf{H}||_F^2$, as already defined before. Making use of (13) and (30), we consequently get

$$P_{\text{out}} = \sum_{\nu=1}^{P_1}\sum_{n=1}^{\rho_{1,\nu}}\sum_{\eta=1}^{P_2}\sum_{l=1}^{\rho_{2,\eta}} \frac{\theta_{1,\nu,n}\,\theta_{2,\eta,l}}{\Gamma(n)\,\Gamma(l)\,\phi_{1,\nu}^n\,\phi_{2,\eta}^l}\cdot\mathcal{I}_1, \quad (31)$$

where we have introduced for brevity the short-hand notation

$$\mathcal{I}_1 = \int_0^{\Upsilon(R)}\int_0^\infty \xi^{l-1}\,x^{n-l-1}\,e^{-\left(\frac{x}{\phi_{1,\nu}}+\frac{\xi}{x\,\phi_{2,\eta}}\right)}\,dx\,d\xi. \quad (32)$$

Changing the order of integration and making use of [11] eq. (3.381,1), we find

$$\mathcal{I}_1 = \int_0^\infty \phi_{2,\eta}^l\,x^{n-1}\,e^{-\frac{x}{\phi_{1,\nu}}}\,\gamma\left(l,\frac{\Upsilon(R)}{x\,\phi_{2,\eta}}\right)dx, \quad (33)$$

where $\gamma(\cdot,\cdot)$ denotes the lower incomplete gamma function [11]. Exploiting the well-known relationship from [11] eq. (8.352,1) that for positive integers n

$$\gamma(n,x) = \Gamma(n)\left(1 - e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{\Gamma(k+1)}\right) \quad (34)$$

with $\Gamma(\cdot)$ as the well-known gamma function [11], we get

$$\mathcal{I}_1 = \Gamma(l)\,\phi_{2,\eta}^l\left[\int_0^\infty x^{n-1}\,e^{-\frac{x}{\phi_{1,\nu}}}\,dx - \sum_{k=0}^{l-1}\left(\frac{\Upsilon(R)}{\phi_{2,\eta}}\right)^k \times \int_0^\infty \frac{x^{n-k-1}\,e^{-\left(\frac{x}{\phi_{1,\nu}}+\frac{\Upsilon(R)}{x\,\phi_{2,\eta}}\right)}}{\Gamma(k+1)}\,dx\right]. \quad (35)$$

This integral finally can be solved in closed-form as

$$\mathcal{I}_1 = \Gamma(l)\,\phi_{2,\eta}^l\left[\phi_{1,\nu}^n\,\Gamma(n) - 2\sum_{k=0}^{l-1}\left(\frac{\Upsilon(R)}{\phi_{2,\eta}}\right)^{\frac{n+k}{2}} \frac{\phi_{1,\nu}^{\frac{n-k}{2}}}{\Gamma(k+1)}\,K_{n-k}\left(2\sqrt{\frac{\Upsilon(R)}{\phi_{1,\nu}\,\phi_{2,\eta}}}\right)\right], \quad (36)$$

where we made use of [11] eq. (3.381,4) and (3.471,9) and with $K_\nu(\cdot)$ as the ν-th order modified Bessel function of the second kind again. Plugging this expression in (31) and recognizing by comparison of coefficients between (9) and (10) that

$$\sum_{\nu=1}^{P_1}\sum_{n=1}^{\rho_{1,\nu}}\sum_{\eta=1}^{P_2}\sum_{l=1}^{\rho_{2,\eta}}\theta_{1,\nu,n}\,\theta_{2,\eta,l} = 1, \quad (37)$$

we finally obtain the result given in Theorem 4. ∎

V. Numerical Results

In the following, we restrict due to space constraints and for simplicity to considering the special but practically important case that $m_{1,i}$ = m_{RX} and $\Omega_{1,i}$ = Ω_{RX} for $1 \leq i \leq N_{RX}$ and similarly that $m_{2,j}$ = m_{TX} and $\Omega_{2,j}$ = Ω_{TX} for $1 \leq j \leq N_{TX}$, i.e., all elements of the SIMO channel $\mathbf{h}_1$ and all elements of the MISO channel $\mathbf{h}_2$ share the same fading levels and average power gains, respectively. Fig. 1 depicts the exact ergodic channel capacity, the upper and lower bounds according to (22) and (18), respectively, as well as the corresponding high SNR asymptotics for several different antenna configurations with the same number of antennas at both sides, i.e., N_{TX} = N_{RX} = N, unity average power gains, and m_{TX} = m_{RX} = 2. First of all, it can be seen there is basically a perfect match between simulated and calculated values, what verifies the validity of our analytical results. Furthermore, it can be seen that in the high SNR regime our bounds are really tight and particularly the lower bound according to (18) is even for low to moderate average SNRs hardly distinguishable from the exact capacity curves. Finally, it is quite obvious from Fig. 1 that keyhole channels do not provide any spatial multiplexing gain since the gradient of all capacity curves is always the same, independent of the actual antenna configuration. This can also be checked analytically by showing that for arbitrary channel parameters $\lim_{\bar{\gamma}\to\infty}\frac{C_{\text{erg}}}{\log_2\bar{\gamma}}$ = $\lim_{\bar{\gamma}\to\infty}\frac{C_{\text{high}}}{\log_2\bar{\gamma}}$ = 1.

Fig. 2 illustrates the impact of the fading severity and possible fading imbalances on the ergodic capacity of a 2×2 MIMO system with Ω_{TX} = Ω_{RX} = 1 for an average SNR of $\bar{\gamma}$ = 20 dB. Obviously, the impact of the fading level on the ergodic capacity is generally rather small and for $m_{TX}\to\infty$, the capacity is upper-bounded by a value which is dependent on the value of the fading parameter m_{RX}. This is because for $m_{TX}\to\infty$, the magnitudes of the individual elements of $\mathbf{h}_2$ become constant and therefore the capacity of the keyhole channel corresponds to the capacity of a SIMO-channel with channel coefficients having fading levels m_{RX}, what can easily be seen from (4). Similarly, in case that m_{TX} is constant while $m_{RX}\to\infty$, the capacity would be upper-bounded for increasing values of m_{RX} by a value which depends on m_{TX}.

Finally, Fig. 3 shows the information outage probability for N_{TX} = N_{RX} = 2, m_{TX} = 2, R = 2 bits per channel use, unity average power gains, and different values of m_{RX}. As can be seen, for m_{RX} = 2, the outage probability is significantly reduced compared to the case with m_{RX} = 1/2, for example, and the slope of the corresponding curve in the high SNR regime is much steeper, what reflects the increased diversity order due to less severe fading at the receiver-side in that case. However, if m_{RX} is further increased, the additional reduction of the outage probability is comparatively small and the slope of the corresponding curves for large values of $\bar{\gamma}$ particularly remains unchanged. This is because the diversity order of the considered keyhole channel is generally given by $\min\{m_{TX}\, N_{TX}, m_{RX}\, N_{RX}\}$, what directly follows from the results reported in [14], for instance. Hence, for $m_{RX} \geq 2$, the diversity order always equals $N_{TX}\, m_{TX} = 4 \leq N_{RX}\, m_{RX}$.

VI. Conclusions

Exact analytical closed-form expressions for the ergodic capacity and information outage probability of Nakagami-m fading MIMO channels with keyhole have been derived. In this regard, we have considered the most general case, where different channel coefficients do not necessarily have the same fading parameter and average power gain, respectively. The ergodic capacity has been expressed as a finite sum of several weighted Meijer G-functions, which might be easily evaluated numerically using standard mathematical software packages. Nevertheless, we additionally provided asymptotically tight upper and lower bounds, which have been expressed by means of elementary functions only. Numerical results were shown to be in excellent agreement with simulation results and hence verified the accuracy and validity of our theoretical analysis.

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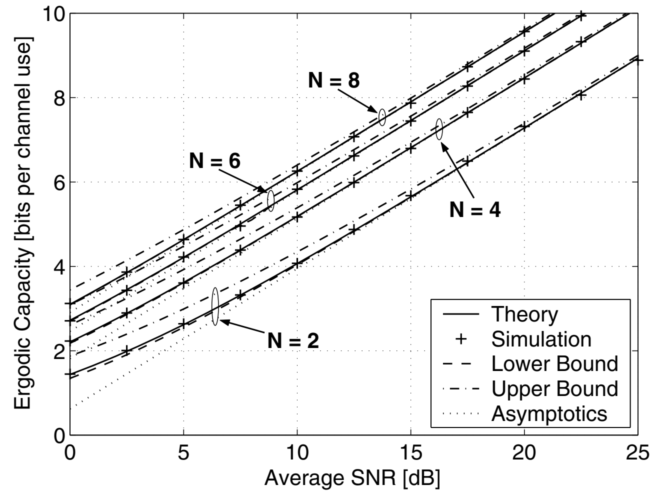

Fig. 1. Ergodic channel capacity for different antenna configurations with $N_{TX} = N_{RX} = N$, $m_{TX} = m_{RX} = 2$, and $\Omega_{TX} = \Omega_{RX} = 1$.

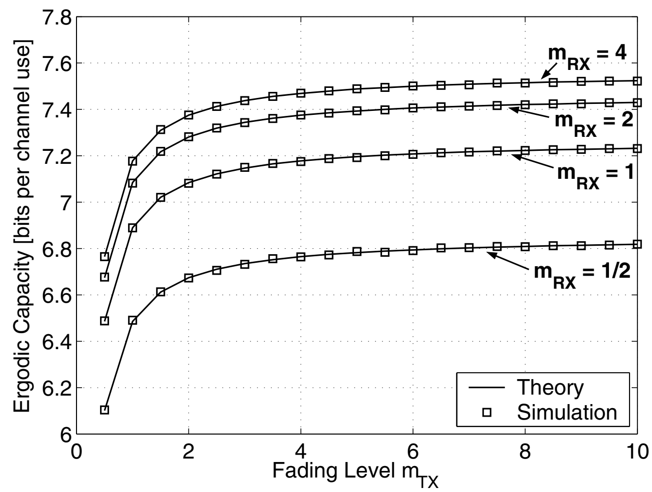

Fig. 2. Ergodic channel capacity for $N_{TX} = N_{RX} = 2$, $\Omega_{TX} = \Omega_{RX} = 1$, $\bar{\gamma} = 20\,\text{dB}$, and different fading levels m_{TX} and m_{RX}.

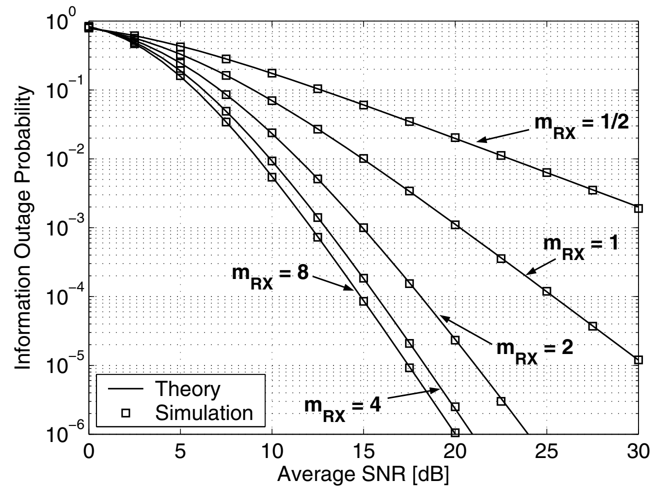

Fig. 3. Information outage probability for $N_{TX} = N_{RX} = 2$, $m_{TX} = 2$, $\Omega_{TX} = \Omega_{RX} = 1$, and $R = 2$ bits per channel use.