Spectrally Concentrated Impulses for Digital Multicarrier Modulation

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Abstract

The design of optimum impulses for digital orthogonal multicarrier Offset-QAM (MC-OQAM) is presented. The impulses are designed such that intercarrier and intersymbol interference are exactly zero and the maximum energy of the impulse is concentrated around the subcarrier. This solution is obtained by solving a nonlinear optimization problem with constraints and by expanding the impulse into a discrete prolate spheroidal sequence. The design method and optimum impulses are presented. With this technique an almost flat spectrum in the pass-band around the subcarrier frequency is achieved, and the out-of-band spectral parts are more than 10 dB below the values of a rectangular impulse, which is used by conventional Orthogonal Frequency Division Multiplexing (OFDM) with Inverse Discrete Fourier Transform (IDFT). Thus, the presented MC-OQAM provides much lower spectral overlap between modulated subcarriers and is more robust against frequency-selective noise. This will be of advantage for channels which suffer from heavy frequency selective interference like the CaTV return channel.

1. Introduction

Multicarrier modulation schemes first appeared in the literature in the late 50's, but only in the last decade they have found widespread use in many applications. OFDM is now being used in many wireless (DAB, DVB-T, radio LAN) as well as in wired applications (ADSL). Its main advantages over single carrier systems are the immunity against impulsive noise and multipath fading and the lack of a complex equalizer.

Most applications of OFDM use DFT-processing and a cyclic prefix as a guard interval to combat intersymbol- (ISI) and intercarrier-interference (ICI). This is a very attractive choice from an implementation point of view. DFT-processing inherently incorporates impulse shaping with a rectangular impulse at the transmitter and the receiver with different filter lengths due to the guard interval. However, the drawbacks are the violation of the matched-filter-criterion at the receiver due to the guard interval and the slowly decreasing spectra of the modulated subcarriers according to a $\sin(f)/f$ function. As a consequence of the second point, the modulated subcarriers exhibit a considerable spectral overlap (while maintaining orthogonality) which leads to the effect that even frequency-selective noise, e.g. amateur radio, not only affects the subcarrier centered at this frequency band, but also – to a certain extend – the adjacent subcarriers. Furthermore, the guard interval reduces the spectral efficiency as the cyclic prefix contains no user data.

During the last years there has been an increasing interest in impulse shaping [1], [2]. By designing proper impulse shapes different from the rectangular impulse, the main spectral parts of the modulated subcarriers can be well localized in the frequency domain, to minimize overlap and sensitivity to frequency-selective interferers.

This paper presents new optimum impulse shapes for digital multicarrier offset-QAM (MC-OQAM). As an optimization criterion, the signal energy in the frequency band around one subcarrier is maximized, resulting in a minimum spectral overlap between all modulated subcarriers.

2. System Model

Our optimization is based on the system model in Fig. 1 which is the baseband equivalent to a MC-OQAM system with N subcarriers. It can be considered as a straightforward extension of a single carrier OQAM.

The incoming symbols $D_\nu[k]$, which may represent one bitstream after serial-to-parallel-conversion, arrive at symbol rate $1/T_S$. They are first upsampled by the factor $M \geq N$ before their real and imaginary parts are filtered with $g[n]$ or $g[n-M/2]$, respectively. For even carrier index ν, the real part of the input symbol $D_\nu[k]$ is fed into an impulse shaping filter with impulse response $g[n]$ and the imaginary part of $D_\nu[k]$ is processed by $g[n-M/2]$, and vice versa for ν odd. As we will see later, the reason for introducing the offset (delay $M/2$) is to reduce the number of constraints for the orthogonality condition by factor one half. The filter output signal is modulated with the complex exponential function $w^{\nu \cdot n}$, where $w = \exp(j2\pi/M)$, giving a subcarrier spacing of $\Delta\omega = 2\pi/T_S$. The number N of subcarriers is always assumed to be even.

In Fig. 1, k denotes the discrete time index at symbol rate $1/T_S$ while n corresponds to the higher sampling rate $1/T_A = M \cdot 1/T_S$. If the upsampling factor M is chosen as an integer multiple of the number of subcarriers N, the multipliers can be integrated into the filter bank. For $M = N$, the system is referred to as critically sampled or

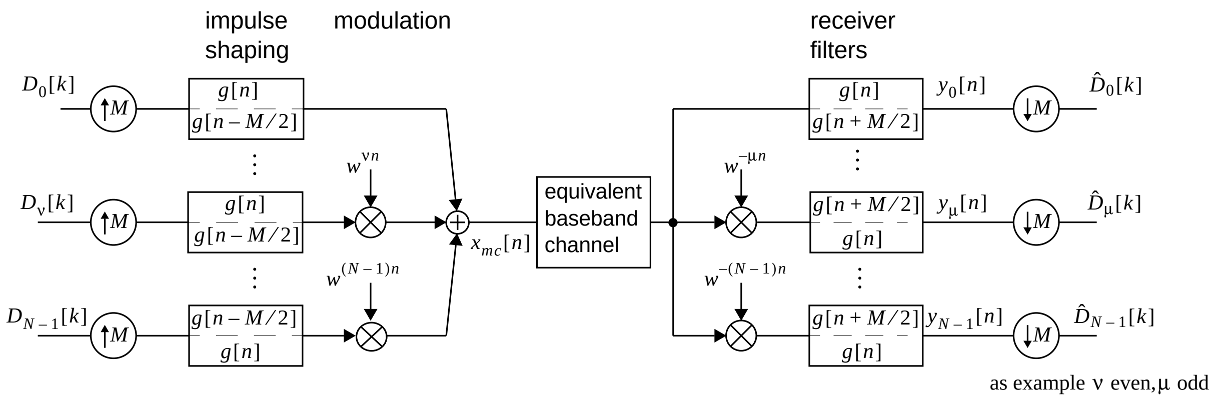

Fig. 1: Block diagram for a digital multicarrier offset-QAM system. The impulse shaping filters alternately delay the real part or the imaginary part of their input signals by half a symbol period.

maximally decimated [3]. Note that the filters are operating at the (faster) sampling rate $1/T_A$. OFDM without pulse shaping, i.e. the well-known implementation with FFT-processing, can be represented in this system model by defining $g[n]$ as rectangular impulse of duration M without delaying the real or the imaginary part. An efficient implementation of an MC-OQAM system using FFT-processing and polyphase filters can be found in [4].

The receiver exhibits a corresponding structure. For the analysis of the intersymbol and interchannel interference we assume an ideal channel. The multicarrier signal for an arbitrary input sequence is given by

$$x_{mc}[n] = \sum_{l=0}^{(N-2)/2} \left\{ w^{2ln} \sum_{k=-\infty}^{\infty} (D'_{2l}[k] + jD''_{2l+1}[k]w^n)g[n-kM] + (D'_{2l+1}[k]w^n + jD''_{2l}[k])g\left[n-\frac{M}{2}-kM\right] \right\} \quad (1)$$

where D'_ν and D''_ν denote the real and the imaginary part of D_ν, respectively.

We now define the elementary impulse $r_{\nu,\mu}[n]$ as the response at the receiver side in branch μ to a transmitted unit impulse in branch ν, [5]:

$$r_{\nu,\mu}[n] = y_\mu[n] \text{ if } D_i[k] = \delta[i-\nu]\cdot\delta[k] \quad (2)$$

$\nu, \mu, i = 0, 1, \ldots, N-1$, $\delta[k]$ is the Kronecker delta.

For the MC-OQAM system the elementary impulses are obtained after some calculation as

$$r_{\nu,\mu}[n] = \sum_{i=-\infty}^{\infty} g\left[n-i-(\nu \mathrm{Mod} 2)\frac{M}{2}\right]\cos\left(\frac{2\pi}{M}d(n-i)\right)g\left[i+(\mu \mathrm{Mod} 2)\frac{M}{2}\right] + j\sum_{i=-\infty}^{\infty} g\left[n-i-(\nu \mathrm{Mod} 2)\frac{M}{2}\right]\sin\left(\frac{2\pi}{M}d(n-i)\right)g\left[i+((\mu+1)\mathrm{Mod} 2)\frac{M}{2}\right] \quad (3)$$

with $d = \nu - \mu$, $\nu, \mu = 0, 1, \ldots, N-1$

Based on these impulses we can now define the condition for zero interference[1]:

$$r_{\nu,\mu}[k\cdot M] = \delta[\nu-\mu, k] \quad (4)$$

with $k \in Z$, $d \in \{-N+1, \ldots, N-1\}$

This condition is often referred to as the extended Nyquist criterion as it not only defines ISI-free but also ICI-free transmission. It is also called the orthogonality condition for OFDM. Equations (4) imposes a condition on the impulse shaping filter $g[n]$ in Eq. (3). To simplify the investigation of Eq. (3), we introduce some assumptions which are convenient for practical filter implementation:

$$g[-n] = g[n] \quad (5)$$

$$g[n] = 0 \text{ for } |n| \geq LM/2, \quad (6)$$

that is, the impulse is even and of length $LM-1$. Inserting (5) into (3), we conclude after some calculation that $Im\{r_{\nu,\mu}[k\cdot M]\} = 0$ holds, i.e. at the sampling instants the elementary impulses take on real values. Thus we get from (3):

$$r_{\nu,\mu}[kM] = \sum_{i=-\infty}^{\infty} g\left[kM-i-(\nu \mathrm{Mod} 2)\frac{M}{2}\right]\cos\left(\frac{2\pi}{M}d\cdot i\right)g\left[i+(\mu \mathrm{Mod} 2)\frac{M}{2}\right] \quad (7)$$

It can be shown, that $r_{\nu,\mu}[kM]$ in (7) satisfies condition (4), if $d = \nu - \mu$ is odd. For $d = 2\chi$ even, we obtain from (7) after some calculation:

$$r[k,\chi] = r_{\nu,\mu}[k\cdot M] = \sum_{n=-\infty}^{\infty} g\left[n-\frac{kM}{2}\right]g\left[n+\frac{kM}{2}\right]\cos\left(\frac{4\pi}{M}\chi n\right) \stackrel{!}{=} \delta[\chi, k] \quad (8)$$

The index range $\chi \in \{0, \ldots, N/2-1\}$ is sufficient, as both sides of (8) are even in χ.

Now, we make use of assumption (6). The time-shifted

1. Strictly speaking, we would have to define an analogous condition for the imaginary part, but it can be shown that for the above system model both conditions are equivalent.

impulses $g[n-kM/2]$ and $g[n+kM/2]$ in (8) overlap as a function of n for $-L<k<L$, and as both sides of (8) are even in k due to (5), it is sufficient to consider only the range $k = 0, \ldots, L-1$. Thus, we obtain from (8)

$$r[k,\chi] = \sum_{n=-(L-k)M/2+1}^{(L-k)M/2-1} g\left[n-\frac{kM}{2}\right] g\left[n+\frac{kM}{2}\right] \cos\left(\frac{4\pi}{M}\chi n\right) \overset{!}{=} \delta[\chi,k] \tag{9}$$

with $\chi = 0, \ldots, N/2-1$ and $k = 0, \ldots, L-1$

(9) is an equation system with $LN/2$ equations[2] and $LM/2$ independent variables, namely $g[0], \ldots, g[LM/2-1]$. In a critically sampled multicarrier system ($M = N$), as it is the case with most of today's OFDM systems, we find that the number of independent variables equals the number of constraints (9), i.e. there is no degree of freedom left for impulse shaping. In order to get some freedom for designing the spectral properties of $g[n]$ we have to increase the sampling rate. As for practical reasons, it is convenient (although not necessary) that the upsampling factor be an integer multiple of the number of carriers. For the following we choose

$$M = 2 \cdot N \tag{10}$$

As mentioned earlier, OFDM systems with DFT-processing use critical sampling and therefore they have to leave a certain number of subcarriers unmodulated in order to avoid aliasing. In contrast to that, in our system in Fig. 1, all subcarriers can be used. Please note that the spectral efficiency is not affected by the choice of M as the required channel bandwidth is given by $\omega_B = N \cdot \Delta\omega = N \cdot 2\pi/T_S$, independent of M.

3. Concentration of the Spectrum

In order to achieve a good spectral concentration of the modulated subcarriers, it is desirable to have as much signal energy as possible inside the frequency band

$$|\omega| \le \eta\frac{\Delta\omega}{2} \tag{11}$$

where $\Delta\omega = 2\pi/T_S$ denotes the subcarrier spacing. Ideally, for $\eta = 1$, *all* signal energy would be inside the frequency band and there would be no spectral overlap between adjacent modulated subcarriers. For $\eta = 2$, each modulated subcarrier would only overlap with its two neighbors. Both cases can be realized only if we allow for an infinite impulse length. For $\eta = 1$ the solution is a rectangular spectrum with the corresponding time function

$$g(t) = \frac{1}{T_S}\operatorname{sinc}\left(\frac{\pi t}{T_S}\right), \text{ with } \operatorname{sinc}(x) = \begin{cases} \sin(x)/x & \text{for } x \ne 1 \\ 1 & \text{for } x = 1 \end{cases}$$

2. Without the offset, at this point we would have twice as much equations.

and for $\eta = 2$ we could use e.g. the well-known square-root-raised-cosine (srrc) impulse with roll-off-factor $\alpha = 1$:

$$g(t) = \frac{4}{\pi T_S}\frac{\cos(2\pi t/T_S)}{1-(4t/T_S)^2}$$

In [1] an optimization procedure was developed for an analog system model which achieves optimal impulses for $\eta = 2$ and an impulse length of $2T_S$ and $4T_S$. In this paper we focus on a digital implementation and an optimization for one subcarrier interval ($\eta = 1$). The signal energy inside the frequency band defined by (11) is given by

$$\begin{aligned} E_\eta &= \frac{T_A}{2\pi}\int_{-\eta\Delta\omega/2}^{\eta\Delta\omega/2} \left|G(e^{j\omega T_A})\right|^2 d\omega = \\ &\frac{\eta}{2N}\sum_{n=-LN+1}^{LN-1} g[n] \sum_{m=-LN+1}^{LN-1} g[m]\operatorname{sinc}\left(\frac{\eta\pi}{2N}(n-m)\right) \end{aligned} \tag{12}$$

where $G(z)$ denotes the double-sided z-transform of $g[n]$ and the second term is obtained by applying Parselval's theorem. E_η has to be maximized with respect to $g[n]$, $n = 0, \ldots, LN-1$ under the constraints (9).

3.1 Discrete Prolate Spheroidal Sequences

For the maximization of the signal energy E_η we expand the impulse $g[n]$ into a series of indexlimited sequences. For this purpose the so called discrete prolate spheroidal sequences (dpss) [6], [7], also named Slepian sequences, are especially well suited because they are the set of index-limited orthogonal sequences with the most concentrated spectra. The dpss $v_k[n]$ are defined by

$$2W\sum_{m=0}^{N_p-1}\operatorname{sinc}(2\pi W(n-m))v_k[m] = \lambda_k v_k[n] \tag{13}$$

with $n \in Z, k \in \{0, \ldots, N_p-1\}$, and depend on the parameters N_p and W.

The λ_k are the eigenvalues for which holds:

$$1 \ge \lambda_0 \ge \lambda_1 \ge \ldots \ge \lambda_{N_p-1} > 0 \text{, and}$$

$$\sum_{k=0}^{N_p-1}\lambda_k = 2WN_p \tag{14}$$

The dpss show the interesting property that they are orthogonal on both the interval $0, \ldots, N_p-1$ and on $-\infty, \ldots, \infty$:

$$\sum_{n=0}^{N_p-1} v_i[n]\cdot v_j[n] = \lambda_i\sum_{n=-\infty}^{\infty} v_i[n]\cdot v_j[n] = \delta[i-j] \tag{15}$$

The dpss are alternately symmetric and antisymmetric with respect to $n = (N_p-1)/2$, i.e. the sequences with even index show an even symmetry while the sequences with odd index exhibit an odd symmetry:

$$v_k[n] = (-1)^k v_k[N_p - 1 - n] \quad (16)$$

We can now expand the impulse $g[n]$ into a series with the dpss as basis functions, taking into account the symmetry properties of both sequences.

$$g_1[n] = g[n - LN] = \sum_{i=0}^{LN} a_i v_{2i}[n] \quad (17)$$

$$\text{with } a_i = \sum_{n=0}^{2LN} g_1[n] v_{2i}[n] \quad \text{and} \quad N_p = 2LN + 1 \quad (18)$$

Putting (17) into (12), the signal energy as a function of the expansion coefficients a_i becomes

$$E_\eta = \frac{\eta}{2N} \sum_{i=0}^{LN} a_i \sum_{j=0}^{LN} a_j \sum_{n=0}^{2LN} v_{2i}[n] \sum_{m=0}^{2LN} v_{2j}[m] \operatorname{sinc}\left(\frac{\eta\pi}{2N}(n-m)\right)$$

Using the definition (13) of the dpss with $W = \eta/(4N)$ and the orthogonality relation (15) we obtain

$$E_\eta = \sum_{i=0}^{LN} a_i^2 \lambda_{2i} \quad (19)$$

The constraints (9), expressed with the expansion coefficients, are

$$r[k, \chi] = \sum_{i=0}^{LN} \sum_{j=0}^{LN} a_i a_j \sum_{n=kN}^{(2L-k)N} v_{2i}[n - kN] v_{2j}[n + kN] \cos\left(\frac{2\pi}{N}\chi n\right) \overset{!}{=} \delta[\chi, k] \quad (20)$$

Equation (19) and (20) form a nonlinear optimization problem for the expansion coefficients a_i: the signal energy (19) has to be maximized under the condition that the constraints (20) have to be satisfied. This can be done by a numerical procedure [8] which uses a sequential quadratic programming (SQP) method.

4. Numerical Solution

The numerical optimization process results in the expansion coefficients a_i as shown in Fig. 2. Except for the first few coefficients, all a_i have values close to zero. This is intuitively clear because the first dpss $v_0[n]$ has the most concentrated spectrum, the next dpss $v_1[n]$ is the sequence with the most concentrated spectrum that is orthogonal to $v_0[n]$, etc.

The resulting elementary impulses $r_{\nu,\mu}[n]$ in Fig. 3 clearly fulfill the extended Nyquist criterion: at the sampling instants except for $\nu = \mu, n = 0$ all samples are zero.

Fig. 4 shows the absolute value of the spectrum $G(e^{j\omega T_A})$ of the optimized impulse versus the spectrum of the conventional rectangular impulse used by OFDM with DFT. In

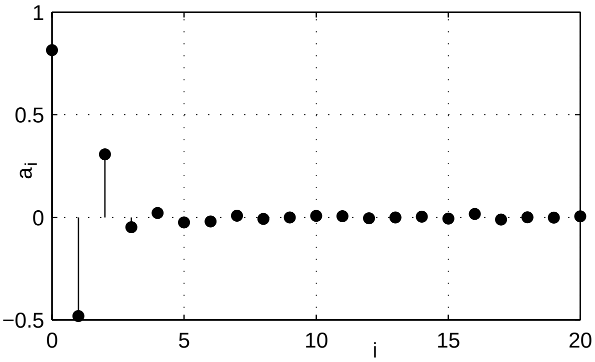

Fig. 2: Expansion coefficients after optimization. The coefficients not shown here are all close to zero.

the interval $-\Delta\omega/2 < \omega < \Delta\omega/2$, 91.3% of the signal energy of the optimum impulse is concentrated and the spectrum is nearly flat, while outside it decreases more rapidly, resulting in less spectral overlap between the modulated subcarriers.

In order to compare our solution with other known impulse shapes, we define the interference power which includes ISI and ICI and corresponds to the squared error with respect to the extended Nyquist criterion:

$$e = \sum_{l=1}^{LN/2-1} c_l^2 \quad \text{with } c_l = r[k, \chi], \; l = k \cdot \frac{N}{2} + \chi \quad (21)$$

$$\text{where } k = l \text{ Div } \frac{N}{2} \text{ and } \chi = l \text{ Mod } \frac{N}{2}$$

Table 1 compares the interference power and the in-band-energy for several known impulses and for the new optimized impulse. Shown are the values for $N = 128$ subcarriers and for an impulse length of 4 symbol intervals, i.e. $L = 4$.

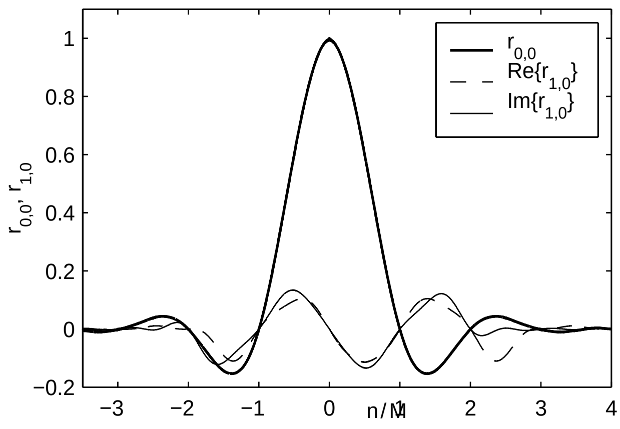

Fig. 3: Elementary impulses for the optimized pulse shape.

Table 1: Interference powers and signal energy concentrations for different pulse shapes. All impulses have normalized signal energy 1. E_η according to (12).

impulse	interference power e	E_1	E_2
optimized impulse (for $\eta = 1$)	-129.55 dB	91.30 %	98.87 %
sinc (truncated)	-14.76 dB	97.80 %	99.97 %
truncated square root raised cosine (srrc)	-38.22 dB	81.72 %	99.98 %
gaussian	-13.64 dB	92.37 %	99.96 %
rectangular	-30.14 dB[a]	77.21%	90.28 %

a. The guard interval was not considered for this calculation. With guard interval the interference power will be much lower.

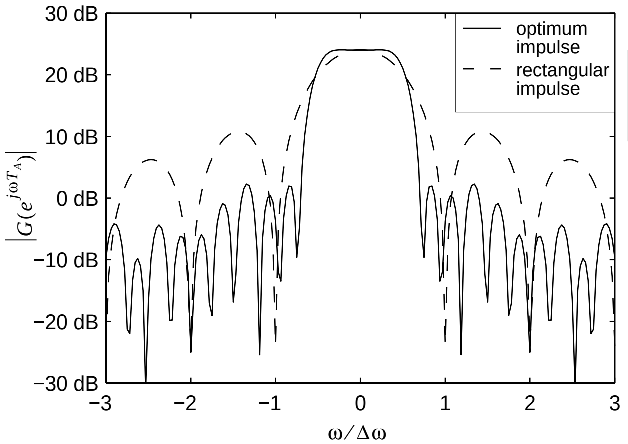

Fig. 4: Magnitude of the spectrum $|G(e^{j\omega T_A})|$ of the optimized impulse in comparison with the rectangular impulse.

From the values in Table 1 we can draw the following conclusions:

- The optimized impulse achieves by far the smallest interference power and a much better energy concentration than the rectangular or the srrc-impulse. Of course, the energy concentration of the truncated sinc impulse cannot be reached as this would not be in accordance with the orthogonality condition.
- The untruncated sinc-impulse concentrates all its signal energy within $|\omega| \le \Delta\omega/2$, but the truncation leads to an increase of the interference power.
- As we can observe from the last column, the truncated square-root-raised-cosine impulse has practically all its signal energy in the range $|\omega| < \Delta\omega$. Therefore, and because of the relatively low interference power, for $\eta = 2$ (but not for $\eta = 1$) this impulse would be already a feasible solution.
- The gaussian impulse $g(t) = \exp(-\pi(t/T_s)^2)$ has some interesting mathematical properties like its invariance with respect to the Fourier transform and the very rapid decrease of its spectrum. However, even the untruncated impulse does not satisfy the Nyquist criterion. Thus, even for a flat channel, powerful equalization has to be used [9].

5. Conclusion

A procedure for designing new impulses for a digital orthogonal MC-OQAM system has been presented. The impulses obtained maximize the signal energy inside the frequency band around the subcarrier, resulting in minimal spectral overlap between the subcarriers. The new impulses provide an almost flat spectrum in the pass-band around the subcarriers and the out-of-band spectral parts are more than 10 dB below the values of a conventional OFDM with IDFT. Also, the interference power due to ICI and ISI is much lower with the new impulses. These properties are of great advantage, if frequency-selective noise is present on the channel or for multi-user applications, where spectral overlap between different users has to be minimized. The MC-OQAM requires no guard interval, so the spectral efficiency compared to the conventional OFDM is increased.

6. References

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