MIMO Zero-Forcing Receivers Part I: Multivariate Statistical Analysis

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Abstract

In this paper, we analyze the signal to noise ratio (SNR) statistics of a multiple input multiple output (MIMO) zero-forcing (ZF) receiver in a correlated Rayleigh fading environment. We present a novel mathematical approach based on multivariate complex Gaussian integrals that enables us for the first time to calculate the moment generating function (MGF) and probability distribution function (PDF) for arbitrary fading correlation at receive and transmit antenna arrays in closed form. It is demonstrated that the MGF can be expressed in terms of the expected value of a ratio of determinants of complex matrix Gaussian random quadratic forms. To the authors' best knowledge, we calculate for the first time closed form expressions for this expected value. Interestingly, we obtain concise formulas for MGF and PDF in terms of certain elementary symmetric functions of the eigenvalues of the MIMO channel correlation matrices. Based on the MGF and PDF, we calculate closed form SER expressions for arbitrary quadrature amplitude modulation (QAM) constellations and present results on mean mutual information. All results are exact and non-asymptotic. The new mathematical techniques presented in this paper have a general scope and can be applied for solving other problems in information theory, for example the performance analysis of MIMO minimum mean squared error receivers.

Index Terms

MIMO, ZF, zero-forcing receiver, multivariate statistics, quadratic forms, complex Gaussian

I. Introduction

Research on the performance analysis of wireless MIMO systems in the majority of cases focuses on Shannon capacity (in particular ergodic capacity) and pairwise error probability (PEP)

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for maximum likelihood receivers. While ergodic capacity [1] [2] [3] [4] [5] [6] and PEP [7] [8] are well understood, only little is known about the symbol error rate (SER) performance of low-complexity linear MIMO receivers, especially in the presence of fading correlation at the receive antenna array. For uncorrelated Rayleigh fading, it was shown in [9] in the context of smart antenna systems that for zero-forcing (ZF) receivers, the subchannel signal to noise ratio (SNR) (for each user) follows a simple gamma distribution. This result was extended for MIMO systems to cover the case of fading correlation at the transmit antenna array in [10] and independently in [11]. On the other hand, many results are available on the analysis of minimum mean squared error (MMSE) processing (which is termed optimum combining in smart antenna literature) with spatially uncorrelated fading. The exact subchannel SINR distribution for users with different transmit powers was given in [12] based on a statistical result on certain matrix quadratic forms in [13]. For equal-power interferers, an exact SER analysis was presented in [14], where the eigenvalue probability density function of complex Wishart matrices was used for the derivation [15]. However, to the authors' best knowledge, no general exact analytical SER expressions can be found in literature for the case of spatial fading correlation at the receive antenna array. Available results for MMSE receivers are approximations or are semi-analytic [16], thus still requiring lengthy Monte-Carlo simulations. For the special case of only two transmit and two receive antennas, exact SER formulas were given in [11] for ZF receivers and in [17] for MMSE receivers based on a random eigenvalue approach for systems with receive as well as transmit correlation. However, these results could not be generalized for an arbitrary number of transmit and receive antennas. In this paper, for the first time we present fully analytic SER expressions for MIMO ZF receivers and an arbitrary finite number of transmit and receive antennas with arbitrary fading correlation at the transmit as well as the receive antenna array. We emphasize that correlation at the receiver (a practically relevant case also in multi-user beamforming scenarios) can be taken into account, which is not possible with other mathematical approaches. In the course of the derivation, we present expressions for the subchannel SNR moment generating function (MGF) in terms of certain expected values of ratios of random determinants. As it appears that there are no results available in literature for calculating these expected values, we present closed form formulas that are derived by a novel mathematical approach. Specifically, we make use of certain complex Gaussian integrals [6] [18] for the derivation. Based on the MGF, we derive exact formulas for arbitrary moments as well as closed form expressions for PDF and

CDF. We show that the SER of ZF receivers in the presence of correlated fading at transmit and receive antenna array can be given in closed form for arbitrary square QAM constellations by using a well-known integral representation of the Gaussian Q function [19]. Moreover, we calculate exact formulas for the mean mutual information (MMI) of the subchannels. The details of the SER and MMI derivations are given in part II of this paper, where we also present novel asymptotical SER expressions for the high SNR regime, which allow for a simple assessment of the influence of the various system parameters and especially fading correlation on the SER performance. Finally, Monte-Carlo simulations for different propagation environments show that the novel SER and MMI formulas exhibit a perfect match.

II. Notation and System Model

A. Notation

Vectors are denoted by bold lowercase letters $\mathbf{x}$, matrices by bold uppercase letters $\mathbf{X}$. Conjugation is indicated by $\mathbf{X}^*$, transposition by $\mathbf{X}^{\mathrm{T}}$ and complex conjugate transpose (Hermitian) by $\mathbf{X}^{\mathrm{H}}$. An identity matrix of size $n \times n$ is written as $\mathbf{I}_n$ and $\mathrm{diag}(x_1, x_2, \ldots, x_n)$ or $\mathrm{diag}(\mathbf{x})$, respectively, returns a diagonal matrix with elements x_k on the diagonal. Equivalently, $\mathrm{diag}(\mathbf{X})$ returns the vector of diagonal elements of square matrix $\mathbf{X}$. The trace of a matrix is denoted by $\mathrm{tr}(\mathbf{X})$. For brevity, we define $\mathrm{etr}(\mathbf{X}) = \exp(\mathrm{tr}(\mathbf{X}))$. The matrix variate complex normal distribution with mean $\mathbf{M}$, m rows and n columns, covariance matrix of column vectors $\mathbf{\Sigma}$, and covariance matrix of row vectors $\mathbf{\Psi}$ is written as $\mathscr{N}_{m,n}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{\Psi})$. By $\sim$ we denote 'is distributed as' and $\simeq$ means 'has the same distribution as'. $\mathbf{X}^\dagger$ is the pseudo-inverse, and the Kronecker product is denoted by $\otimes$. The expected value of a function $f(\mathbf{X})$ with respect to $\mathbf{X}$ has the representation $E_{\mathbf{X}}[f(\mathbf{X})]$. We use the notation $\hat{\alpha}_k$ for index subsets of cardinality $|\hat{\alpha}_k| = k$ (the cardinality can be omitted), complementary index subsets are written as $\hat{\gamma} = \hat{\alpha} \backslash \hat{\beta}$. For example $\hat{\alpha}_3 = \{1,3,5\}$ with $\hat{\beta}_2 = \{1,\ldots,5\} \backslash \hat{\alpha}_3 = \{2,4\}$. By $\{\mathbf{X}\}_{\hat{\beta}}^{\hat{\alpha}}$ we denote a matrix that results from selecting the row subset $\hat{\alpha}$ and the column subset $\hat{\beta}$ from matrix $\mathbf{X}$. Similarly, we let $|\mathbf{X}|_{\hat{\beta}}^{\hat{\alpha}} = \left|\{\mathbf{X}\}_{\hat{\beta}}^{\hat{\alpha}}\right|$, where $|\mathbf{X}|$ denotes the determinant of square matrix $\mathbf{X}$. We make frequent use of elementary symmetric functions of matrix argument with the definition

$$\mathrm{tr}_k(\mathbf{X}) = \sum_{\hat{\alpha}_k} |\mathbf{X}|_{\hat{\alpha}_k}^{\hat{\alpha}_k}, \tag{1}$$

for square $m \times m$ matrix $\mathbf{X}$, where the sum is over all $\binom{k}{m}$ different index subsets of cardinality k. Note that for vector $\mathbf{d} = (d_1, d_2, \ldots, d_m)^T$ and diagonal $m \times m$ matrix $\mathbf{D} = \mathrm{diag}(\mathbf{d})$ the elementary symmetric functions of matrix argument reduce to scalar elementary symmetric functions (with indices $\{i_1, \ldots, i_k\}$)

$$\mathrm{tr}_k(\mathbf{D}) = \mathrm{tr}_k(\mathbf{d}) = \sum_{\{i_1,\ldots,i_k\}} d_{i_1} \cdot \cdots \cdot d_{i_k} \tag{2}$$

with the definition $\mathrm{tr}_0(\mathbf{D}) = 1$ and $\mathrm{tr}_k(\mathbf{D}) = 0$ for all $k < 0$. Note that in (2) the sum is again over all $\binom{k}{m}$ index subsets of cardinality k. For brevity we introduce the notation

$$\mathrm{tr}_k^{(i)}(\mathbf{D}) = \mathrm{tr}_k(\mathrm{diag}(d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_m)). \tag{3}$$

The complete symmetric function $\mathrm{h}_k(\mathbf{x})$ with $\mathbf{x} = (x_1 \ldots, x_n)^T$ is the sum of all monomials m_λ of total degree k in the variables $x_1, x_2, \ldots, x_n$ so that [20]

$$\mathrm{h}_k(\mathbf{x}) = \sum_{|\lambda|=k} m_\lambda \tag{4}$$

with $\mathrm{h}_0(\mathbf{x}) = 1$, $\mathrm{h}_k(\mathbf{x}) = 0$ for all $k < 0$, and $\mathrm{h}_1(\mathbf{x}) = \mathrm{tr}_1(\mathbf{x})$. For example, we have

$$\mathrm{h}_2((x_1, x_2)) = x_1^2 + x_1 x_2 + x_2^2.$$

There is a close relation between complete and elementary symmetric functions [20], namely for a $n \times 1$ vector $\mathbf{x}$

$$\mathrm{tr}_k(\mathbf{x}) = \left|\{\mathrm{h}_{1-i+j}(\mathbf{x})\}\right|_{1 \le i,j \le k} \tag{5}$$

and equivalently

$$\mathrm{h}_k(\mathbf{x}) = \left|\{\mathrm{tr}_{1-i+j}(\mathbf{x})\}\right|_{1 \le i,j \le k}. \tag{6}$$

In (5) and (6) i and j are the row and column index, respectively, of the $k \times k$ matrices. For brevity we introduce the following (normalized) complex matrix differential for complex $M \times N$ matrix $\mathbf{X}$

$$\mathrm{D}_c\mathbf{X} = \prod_{m=1}^{M}\prod_{n=1}^{N}\frac{\Re\{\mathrm{d}x_{11}\}\Im\{\mathrm{d}x_{11}\}\cdots\Re\{\mathrm{d}x_{mn}\}\Im\{\mathrm{d}x_{mn}\}}{\pi} \tag{7}$$

and write for the multidimensional integral

$$\int f(\mathbf{X})\ \mathrm{D}_c\mathbf{X} = \underset{\substack{\Re\{x_{11}\}\\ \Im\{x_{11}\}}}{\iint}\cdots\underset{\substack{\Re\{x_{mn}\}\\ \Im\{x_{mn}\}}}{\iint} f(\mathbf{X})\ \mathrm{D}_c\mathbf{X}, \tag{8}$$

where each scalar integral is over the range $-\infty$ to $+\infty$. Throughout the paper we use the definitions for $m\times 1$ vector $\mathbf{x} = (x_1,\ldots,x_m)^T$

$$K_{\mathbf{x}}(l) = \frac{1}{\prod_{n=1,n\neq l}^{m}(x_l - x_n)} \tag{9}$$

and

$$\tilde{K}_{\mathbf{x}}(\alpha_1,\alpha_2) = \frac{1}{x_{\alpha_1} - x_{\alpha_2}} \tag{10}$$

with the relation

$$x_{\alpha_1}\cdot\tilde{K}_{\mathbf{x}}(\alpha_1,\alpha_2) + x_{\alpha_2}\cdot\tilde{K}_{\mathbf{x}}(\alpha_2,\alpha_1) = 1. \tag{11}$$

B. System Model

We consider a flat fading MIMO link with T transmit and R receive antennas (see Fig. 1), whereas the $R\times T$ channel matrix is given by $\mathbf{H}$. There are L independent data channels and the transmit symbols are arranged in a $L\times 1$ vector $\mathbf{s}$. Furthermore, we introduce a linear $T\times L$ transmit filter matrix $\mathbf{F}$, which maps L subchannels on the T transmit antennas. In general we assume $L\leq T$. On the receiver side we assume without loss of generality (w.l.o.g.) additive white Gaussian noise (AWGN) modeled by the $R\times 1$ vector $\mathbf{n}$ and the $R\times 1$ noisy received vector is denoted by $\mathbf{y}$. Colored noise can be taken into account via a modified receive correlation matrix (see also below). The transmission over the MIMO channel with transmit prefiltering can than be described by

$$\mathbf{y} = \mathbf{HFs} + \mathbf{n} = \mathbf{Ks} + \mathbf{n} \tag{12}$$

with the $R \times L$ compound channel matrix $\mathbf{K} = \mathbf{HF}$. At the receiver side, the received vector $\mathbf{y}$ is processed by the zero forcing (ZF) matrix $\mathbf{G}$ and the $L \times 1$ vector $\mathbf{z}$ results

$$\mathbf{z} = \mathbf{Gy}. \tag{13}$$

The zero-forcing receiver has the well known [10] pseudo inverse receiver matrix

$$\mathbf{G} = \left(\mathbf{K}^{\mathrm{H}}\mathbf{K}\right)^{-1}\mathbf{K}^{\mathrm{H}} = \mathbf{K}^{\dagger}. \tag{14}$$

Finally, we define the diversity of the system by

$$\mathscr{D} = R - L + 1. \tag{15}$$

C. Statistics

In this paper, we investigate the transmission over a Rayleigh fading MIMO link, i.e. the channel matrix $\mathbf{H}$ is complex Gaussian distributed

$$\mathbf{H} \sim \mathscr{N}_{R,T}(\mathbf{0}, \mathbf{R}_{\mathrm{RX}}, \mathbf{R}_{\mathrm{TX}}). \tag{16}$$

Without loss of generality, we assume full rank $\mathbf{R}_{\mathrm{RX}}$ and $\mathbf{R}_{\mathrm{TX}}$. Rank deficient correlation matrices can be mapped on an equivalent system with full rank transmit and receive correlation matrices with a smaller virtual number of transmit and receive antennas, respectively. We note that (16) is the well known [21] [22] MIMO channel model with separable correlation matrices at transmitter $\mathbf{R}_{\mathrm{TX}}$ and receiver $\mathbf{R}_{\mathrm{RX}}$ and

$$\mathbf{H} \simeq \mathbf{A}^{\mathrm{H}}\mathbf{H}_{w}\mathbf{B}, \tag{17}$$

where

$$\mathbf{R}_{\mathrm{RX}} = \mathbf{A}^{\mathrm{H}}\mathbf{A} \tag{18}$$

$$\mathbf{R}_{\mathrm{TX}} = \mathbf{B}^{\mathrm{H}}\mathbf{B}. \tag{19}$$

The Rayleigh fading channel model can further be generalized by allowing for arbitrary variances of the individual channel matrix elements. However, (17) is a good tradeoff between complexity and accuracy. For later reference in subsequent derivations, we introduce the eigenvalue decomposition (EVD) of the receive correlation matrix with diagonal

$$\mathbf{O} = \mathrm{diag}\left(\mathbf{o}\right) = \mathrm{diag}\left(o_1, \ldots, o_R\right), \tag{20}$$

namely

$$\mathbf{R}_{\mathrm{RX}} \quad = \mathbf{V}_{\mathrm{r}}\mathbf{O}\mathbf{V}_{\mathrm{r}}^{\mathrm{H}} \tag{21}$$

with unitary matrix $\mathbf{V}_{\mathrm{r}}$. Straightforward considerations lead to the distribution of the compound channel

$$\mathbf{K} \sim \mathcal{N}_{R,L}(\mathbf{0}, \mathbf{R}_{\mathrm{RX}}, \mathbf{C}) \tag{22}$$

with the equivalent $L \times L$ covariance matrix $\mathbf{C}$, which comprises the effects of transmit correlation as well as transmit prefiltering

$$\mathbf{C} = \mathbf{F}^{\mathrm{H}}\mathbf{R}_{\mathrm{TX}}\mathbf{F}. \tag{23}$$

Throughout this paper we use the definition

$$\left(c^{11}, \ldots, c^{LL}\right)^{T} = \mathrm{diag}\left(\mathbf{C}^{-1}\right). \tag{24}$$

The complex Gaussian pdf of $\mathbf{K}$ is given by [23]

$$p_{\mathbf{K}}(\mathbf{K}) = \frac{1}{\pi^{RL}|\mathbf{C}|^{R}|\mathbf{R}_{\mathrm{RX}}|^{L}} \cdot \mathrm{etr}\left(-\mathbf{C}^{-1}\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right). \tag{25}$$

Without loss of generality we assume white transmit symbols with covariance

$$\mathbf{R}_{\mathrm{ss}} = E_s \cdot \mathbf{I}_L, \tag{26}$$

where E_s is the energy per transmit symbol. Other transmit covariance matrices $\mathbf{R}_{\mathrm{ss}}$ can easily be absorbed in a modified transmit correlation matrix. Equivalently, w.l.o.g. we consider AWGN with covariance

$$\mathbf{R}_{\mathrm{nn}} = N_0 \cdot \mathbf{I}_R, \tag{27}$$

where N_0 is the noise variance per receive antenna. Colored noise can be taken into account by straightforward absorption in the receive correlation matrix. Finally, in the following the mean SNR is defined by

$$\gamma = \frac{E_s}{N_0} \tag{28}$$

and will be used consistently throughout the paper. Furthermore, we use the scaled mean SNR

$$\tilde{\gamma}_k = \frac{\gamma}{c^{kk}} \tag{29}$$

and introduce the scaled vector of eigenvalues

$$\tilde{\mathbf{o}} = (\tilde{o}_1, \ldots, \tilde{o}_R)^T = \tilde{\gamma}_k \cdot \mathbf{o}. \tag{30}$$

for brevity in later derivations. Throughout this paper we use the subchannel index k.

III. SNR Expressions

After splitting the vector $\mathbf{z}$ at the output of the receive filter $\mathbf{G}$ in a signal component $\mathbf{z}_{\mathrm{s}}$ and a noise component $\mathbf{z}_{\mathrm{n}}$

$$\mathbf{z} = \mathbf{GKs} + \mathbf{Gn} = \mathbf{z}_{\mathrm{s}} + \mathbf{z}_{\mathrm{n}}, \tag{31}$$

it can be shown that by the zero forcing property

$$E\left[\mathbf{z}_{\mathrm{s}}\mathbf{z}_{\mathrm{s}}{}^{\mathrm{H}}\right] = E_s \cdot \mathbf{I}_L \tag{32}$$

and for the noise component

$$E\left[\mathbf{z}_{\mathrm{n}}\mathbf{z}_{\mathrm{n}}^{\mathrm{H}}\right] = N_0 \cdot \mathbf{K}^{\dagger}\left(\mathbf{K}^{\dagger}\right)^{\mathrm{H}}. \tag{33}$$

Therefore, the SNR on subchannel k after receive processing reads

$$\gamma_{\mathrm{SC},k} = \frac{\gamma}{\left[\mathbf{K}^{\dagger}\left(\mathbf{K}^{\dagger}\right)^{\mathrm{H}}\right]_{kk}} = \frac{\gamma}{\left[\left(\mathbf{K}^{\mathrm{H}}\mathbf{K}\right)^{-1}\right]_{kk}}. \tag{34}$$

The subchannel SNR can be rewritten in terms of a random quadratic form that is later shown to be well suited for a statistical analysis. For simplifying the notation, in the following we first focus on the subchannel with index $k = 1$. It is then a straightforward exercise to generalize the results to an arbitrary subchannel.

We first partition the compound channel

$$\mathbf{K} = \left[\, \mathbf{k}_1 \;\; \tilde{\mathbf{K}} \,\right], \tag{35}$$

where $\mathbf{k}_1$ is a $R \times 1$ column vector and $\tilde{\mathbf{K}}$ is a $R \times (L-1)$ matrix. For rewriting the SNR expression we can exploit the following result on partitioned inverses. Let

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{22} \\ \mathbf{X}_{21} & \mathbf{X}_{22} \end{bmatrix} \tag{36}$$

and

$$\mathbf{X}^{-1} = \begin{bmatrix} \mathbf{X}^{11} & \mathbf{X}^{22} \\ \mathbf{X}^{21} & \mathbf{X}^{22} \end{bmatrix}. \tag{37}$$

It is then well-known that [24]

$$\mathbf{X}^{11} = \left(\mathbf{X}_{11} - \mathbf{X}_{12}\mathbf{X}_{22}^{-1}\mathbf{X}_{21}\right)^{-1} \equiv \left(\mathbf{X}_{11\cdot 2}\right)^{-1}. \tag{38}$$

With the help of (38) it can be shown that the SNR on subchannel 1 is given by

$$\gamma_{\mathrm{SC},1} = \gamma \cdot \mathbf{k}_1^{\mathrm{H}} \left(\mathbf{I}_R - \tilde{\mathbf{K}}\left(\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)^{-1}\tilde{\mathbf{K}}^{\mathrm{H}}\right)\mathbf{k}_1, \tag{39}$$

which is a random quadratic form in complex Gaussian distributed vectors.

IV. Subchannel SNR MGF

In this section we calculate the moment generating function (MGF) for the subchannel SNR, which is the basis for derivations of the PDF and CDF. The MGF also serves as a basis for mean mutual information (MMI) and SER calculations.

The MGF of the subchannel specific (index k) SNR is given by

$$M_k(s) = E_{\mathbf{K}}\left[\exp(-s \cdot \gamma_{\mathrm{SC},k})\right], \tag{40}$$

where the expected value is with respect to the channel statistics. Note that in accordance with common practice in the area of communication theory (e.g. [19]), we talk about the MGF in (40), even though we use a minus sign in the exponent.

A. MIMO Channel Probability Distribution

For later integrations, it is convenient to reformulate the MIMO channel PDF. We partition the covariance matrix $\mathbf{C}$ with scalar c_{11}, $(L-1)\times 1$ vector $\mathbf{c}_{21}$, and $(L-1)\times(L-1)$ matrix $\mathbf{C}_{22}$ as

$$\mathbf{C} = \begin{bmatrix} c_{11} & \mathbf{c}_{21}^{\mathrm{H}} \\ \mathbf{c}_{21} & \mathbf{C}_{22} \end{bmatrix}. \tag{41}$$

Equivalently, we let

$$\mathbf{C}^{-1} = \begin{bmatrix} c^{11} & (\mathbf{c}^{21})^{\mathrm{H}} \\ \mathbf{c}^{21} & \mathbf{C}^{22} \end{bmatrix}. \tag{42}$$

It is now possible to rewrite the exponential term of the channel PDF as

$$\begin{aligned} \mathrm{etr}\left(-\mathbf{C}^{-1}\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right) &= \mathrm{etr}\left(-\left(c^{11}\mathbf{k}_1^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{k}_1 + \mathbf{C}^{22}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\tilde{\mathbf{K}}\right)\right)\cdot\ldots \\ &\quad \mathrm{etr}\left(-\left(((\mathbf{c}^{21})^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{k}_1 + (((\mathbf{c}^{21})^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{k}_1)^{\mathrm{H}}\right)\right), \end{aligned} \tag{43}$$

where $\tilde{\mathbf{K}}$ and $\mathbf{k}_1$ are explicitly visible.

B. Uncorrelated channel without prefilter

For the case of uncorrelated fading, the subchannel SNR statistics of MIMO ZF receivers are well known [10] [25]. Basically, the subchannel SNR in this case can be expressed as the marginal distribution of a complex Wishart matrix, which has been extensively studied in multivariate statistical literature [26] [27] [28]. However, for an introduction of the novel mathematical techniques deployed in this paper, we also consider this simple case. As expected on the basis of symmetry considerations, the statistics are independent of the subchannel index.

Theorem 1: The MGF of the subchannel SNR in case of uncorrelated Rayleigh fading and no prefilter at the transmitter side $\mathbf{F} = \mathbf{I}_T$ is given by

$$M_u(s) \;=\; \frac{1}{(1+s\cdot\gamma)^{\mathscr{D}}} \tag{44}$$

with the obvious diversity of the system $\mathscr{D} = R - L + 1$. This is the MGF of a Gamma distribution with $\mathscr{D}$ degrees of freedom. In case of no transmit correlation, the MGF is not dependent on the subchannel index k.

Proof: The channel PDF in case of uncorrelated fading is from (25) with the help of (43) given by

$$p_{\mathbf{K},u}(\mathbf{K}) = \frac{1}{\pi^{RL}} \cdot \mathrm{etr}\left(-\left(\mathbf{k}_1^{\mathrm{H}}\mathbf{k}_1 + \tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\right). \tag{45}$$

The subchannel SNR MGF is then given by the integral (note again that due to the symmetry of the problem, an arbitrary subchannel may be considered)

$$\begin{aligned} M_u(s) \;=\; & \int \exp\left(-\mathbf{k}_1^{\mathrm{H}}\left((s\cdot\gamma+1)\cdot\mathbf{I}_R - s\cdot\gamma\tilde{\mathbf{K}}\left(\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)^{-1}\tilde{\mathbf{K}}^{\mathrm{H}}\right)\mathbf{k}_1\right)\cdot\ldots \\ & \mathrm{etr}\left(-\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\mathrm{D}_c\mathbf{k}_1\mathrm{D}_c\tilde{\mathbf{K}} \end{aligned} \tag{46}$$

Now carrying out the integral with respect to $\mathbf{k}_1$ using the well known vector variate Gaussian integal (120) in Appendix I we find after some simple manipulations

$$M_u(s) = \frac{1}{(s\cdot\gamma+1)^R}\int \frac{1}{\left|\mathbf{I}_R - \frac{s\cdot\gamma}{(s\cdot\gamma+1)}\tilde{\mathbf{K}}\left(\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)^{-1}\tilde{\mathbf{K}}^{\mathrm{H}}\right|}\cdot \mathrm{etr}\left(-\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\mathrm{D}_c\tilde{\mathbf{K}} \tag{47}$$

$$= \frac{1}{(s\cdot\gamma+1)^R}\int \frac{1}{\left|\mathbf{I}_{L-1} - \frac{s\cdot\gamma}{(s\cdot\gamma+1)}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\left(\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)^{-1}\right|}\cdot \mathrm{etr}\left(-\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\mathrm{D}_c\tilde{\mathbf{K}}, \tag{48}$$

where we have used (for matrices $\mathbf{A}$ and $\mathbf{B}$ of compatible size) [24]

$$|\mathbf{I}+\mathbf{AB}| = |\mathbf{I}+\mathbf{BA}|. \tag{49}$$

After simplifying the determinant expression and carrying out the integral with respect to $\tilde{\mathbf{K}}$ using (125) in Appendix I we finally have proven the theorem. ■

C. Channel with prefilter and transmit correlation

We present a first generalization of the results of the last subsection.

Theorem 2: In case of transmit correlation or the presence of a prefilter and uncorrelated fading at the receive antenna array the subchannel SNR MGF is given by

$$M_{k,\mathrm{TX}}(s) = \frac{1}{\left(1+s\cdot\frac{1}{c^{kk}}\gamma\right)^{\mathscr{D}}} \tag{50}$$

with the diversity of the system $\mathscr{D} = R-L+1$ and $\mathrm{diag}\left(\mathbf{C}^{-1}\right) = \left(c^{11},\ldots,c^{LL}\right)^T$.

Proof: We demonstrate two different proofs of the theorem. First, we consider the expected value (with respect to the channel statistics) of an arbitrary function f of the subchannel SNR (again we consider exemplarily subchannel $k=1$)

$$E\left[f\left(\gamma_{\mathrm{SC},1}\right)\right] = \int f\left(\frac{\gamma}{\left[\left(\mathbf{K}^H\mathbf{K}\right)^{11}\right]}\right)\cdot\frac{1}{|\mathbf{C}|^R|\mathbf{R}_{\mathrm{RX}}|^L}\cdot\mathrm{etr}\left(-\mathbf{C}^{-1}\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right)\ \mathrm{D}_c\mathbf{K}. \tag{51}$$

With the transformation $\mathbf{X} = \mathbf{C}^{-1/2}\mathbf{K}$ (see e.g. [15] [26] [27] for an introduction to matrix variate variable transformations) we obtain

$$E\left[f\left(\gamma_{\mathrm{SC},1}\right)\right] = \int f\left(\frac{\gamma}{\left[\left(\mathbf{C}^{1/2}\mathbf{K}^H\mathbf{K}\mathbf{C}^{1/2}\right)^{11}\right]}\right)\cdot\frac{1}{|\mathbf{R}_{\mathrm{RX}}|^L}\cdot\mathrm{etr}\left(-\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right)\ \mathrm{D}_c\mathbf{K}. \tag{52}$$

This can be written as

$$E\left[f\left(\gamma_{\mathrm{SC},1}\right)\right]=\int f\left(\gamma\cdot\frac{\left|\mathbf{C}^{1/2}\mathbf{K}^{H}\mathbf{K}\mathbf{C}^{1/2}\right|}{\left|\mathbf{C}_{22}^{1/2}\tilde{\mathbf{K}}^{H}\tilde{\mathbf{K}}\mathbf{C}_{22}^{1/2}\right|}\right)\cdot\frac{1}{\left|\mathbf{R}_{\mathrm{RX}}\right|^{L}}\cdot\mathrm{etr}\left(-\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right)\ \mathrm{D}_{c}\mathbf{K}, \tag{53}$$

where we have used [24] for square matrix $\mathbf{X}$

$$x^{11}=\frac{\left|\mathbf{X}_{22}\right|}{\left|\mathbf{X}\right|}. \tag{54}$$

From that it can be seen that the expected value can be reformulated as

$$E\left[f\left(\gamma_{\mathrm{SC},1}\right)\right]=\int f\left(\gamma\cdot\frac{1}{c^{11}}\cdot\frac{\left|\mathbf{K}^{H}\mathbf{K}\right|}{\left|\tilde{\mathbf{K}}^{H}\tilde{\mathbf{K}}\right|}\right)\cdot\frac{1}{\left|\mathbf{R}_{\mathrm{RX}}\right|^{L}}\cdot\mathrm{etr}\left(-\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right)\ \mathrm{D}_{c}\mathbf{K}, \tag{55}$$

and after generalizing for an arbitrary subchannel

$$E\left[f\left(\gamma_{\mathrm{SC},1}\right)\right]=\int f\left(\frac{\tilde{\gamma}_{k}}{\left[\left(\mathbf{K}^{H}\mathbf{K}\right)^{kk}\right]}\right)\cdot\frac{1}{\left|\mathbf{R}_{\mathrm{RX}}\right|^{L}}\cdot\mathrm{etr}\left(-\mathbf{K}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{K}\right)\ \mathrm{D}_{c}\mathbf{K}, \tag{56}$$

with $\tilde{\gamma}_k=\frac{\gamma}{c^{kk}}$ according to (29), i.e. the presence of fading correlation at the transmit antenna array has just a scaling effect on the mean SNR γ.

We now present a second proof that makes extensive use of Gaussian integrals. The channel PDF is from (25) with the help of (43) given by

$$p_{\mathbf{K},\mathrm{TX}}(\mathbf{K})=\frac{1}{\pi^{RL}\cdot\left|\mathbf{C}\right|^{R}}\cdot\mathrm{etr}\left(-\left(c^{11}\mathbf{k}_{1}^{\mathrm{H}}\mathbf{k}_{1}+\left(\mathbf{c}^{21}\right)^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_{1}+\left(\left(\mathbf{c}^{21}\right)^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_{1}\right)^{\mathrm{H}}+\mathbf{C}^{22}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\right). \tag{57}$$

The subchannel SNR MGF for subchannel 1 is then given by the integral

$$\begin{aligned}M_{1,\mathrm{TX}}(s)\ =\ &\frac{1}{\left|\mathbf{C}\right|^{R}}\cdot\int\exp\left(-s\cdot\gamma\cdot\mathbf{k}_{1}^{\mathrm{H}}\left(\mathbf{I}_{R}-\tilde{\mathbf{K}}\left(\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)^{-1}\tilde{\mathbf{K}}^{\mathrm{H}}\right)\mathbf{k}_{1}\right)\cdot \\ &\mathrm{etr}\left(-\left(c^{11}\mathbf{k}_{1}^{\mathrm{H}}\mathbf{k}_{1}+\left(\mathbf{c}^{21}\right)^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_{1}+\left(\left(\mathbf{c}^{21}\right)^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_{1}\right)^{\mathrm{H}}+\mathbf{C}^{22}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\right)\ \mathrm{D}_{c}\mathbf{k}_{1}\mathrm{D}_{c}\tilde{\mathbf{K}}.\end{aligned} \tag{58}$$

Then note that from (123) in Appendix I, which is one of the key formulas for deriving the novel results in this paper, we obtain the import relation

$$\exp\left(-s\cdot\gamma\cdot\mathbf{k}_1^{\mathrm{H}}\left(\mathbf{I}_R-\tilde{\mathbf{K}}\left(\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)^{-1}\tilde{\mathbf{K}}^{\mathrm{H}}\right)\mathbf{k}_1\right)= \quad (59)$$

$$\frac{1}{(s\cdot\gamma)^{L-1}}\cdot\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|\cdot\exp\left(-s\cdot\gamma\cdot\mathbf{k}_1^{\mathrm{H}}\mathbf{k}_1\right)\cdot\int\mathrm{etr}\left(-\left(\mathbf{k}_1^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}+\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_1+\frac{1}{s\cdot\gamma}\cdot\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}\right)\right)\mathrm{D}_c\mathbf{x} \quad ,$$

where we have removed the inverse in the exponent. Using (59) in (58) we get

$$\begin{aligned} M_{1,\mathrm{TX}}(s) \;=\; & \frac{1}{(s\cdot\gamma)^{L-1}}\cdot\frac{1}{|\mathbf{C}|^R}\cdot\int\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|\cdot \\ & \mathrm{etr}\left(-\left(\left(s\cdot\gamma+c^{11}\right)\cdot\mathbf{k}_1^{\mathrm{H}}\mathbf{k}_1+\left(\mathbf{c}^{21}+\mathbf{x}\right)^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_1+\left(\left(\mathbf{c}^{21}+\mathbf{x}\right)^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_1\right)^{\mathrm{H}}\right)\right)\cdot \\ & \mathrm{etr}\left(-\left(\mathbf{C}^{22}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}+\frac{1}{s\cdot\gamma}\cdot\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}\right)\right)\ \mathrm{D}_c\mathbf{k}_1\ \mathrm{D}_c\tilde{\mathbf{K}}\ \mathrm{D}_c\mathbf{x}. \end{aligned} \quad (60)$$

Integrating with respect to $\mathbf{k}_1$ we find after rearranging the exponential

$$\begin{aligned} M_{1,\mathrm{TX}}(s) \;=\; & \frac{1}{(s\cdot\gamma)^{L-1}\cdot(s\cdot\gamma+c^{11})^R}\cdot\frac{1}{|\mathbf{C}|^R}\cdot\int\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|\cdot \\ & \mathrm{etr}\left(-\left(\frac{c^{11}}{s\cdot\gamma\cdot\omega}\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}-\frac{1}{\omega}\cdot(\mathbf{c}^{21})^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}-\frac{1}{\omega}\cdot\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{c}^{21}\right)\right)\cdot \\ & \mathrm{etr}\left(-\left(\mathbf{C}^{22}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}-\frac{1}{s\cdot\gamma+c^{11}}(\mathbf{c}^{21})^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{c}^{21}\right)\right)\ \mathrm{D}_c\tilde{\mathbf{K}}\ \mathrm{D}_c\mathbf{x} \end{aligned} \quad (61)$$

with $\omega=s\cdot\gamma+c^{11}$. Now carrying out the integral with respect to $\mathbf{x}$ we find after simplifying

$$\begin{aligned} M_{1,\mathrm{TX}}(s) \;=\; & \frac{1}{\left(s\cdot\gamma+c^{11}\right)^{R-L+1}(c^{11})^{L-1}}\cdot\frac{1}{|\mathbf{C}|^R}\cdot \\ & \int\mathrm{etr}\left(-\left(\mathbf{C}^{22}-\frac{\mathbf{c}_{21}\left(\mathbf{c}_{21}\right)^{\mathrm{H}}}{c^{11}}\right)\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right)\ \mathrm{D}_c\tilde{\mathbf{K}}. \end{aligned} \quad (62)$$

Finally integrating with respect to $\tilde{\mathbf{K}}$ we get

$$M_{1,\mathrm{TX}}(s) \;=\; \frac{(c^{11})^{R-L+1}}{\left(s\cdot\gamma+c^{11}\right)^{R-L+1}}\cdot\frac{1}{|\mathbf{C}|^R\cdot(c^{11})^R\cdot\left|\mathbf{C}^{22}-\frac{\mathbf{c}_{21}(\mathbf{c}_{21})^{\mathrm{H}}}{c^{11}}\right|^R}. \quad (63)$$

By using the relation for square matrix $\mathbf{X}$ [24]

$$\left|\mathbf{X}^{-1}\right|=\left|\mathbf{X}^{11}\right|\left|\mathbf{X}^{22}-\mathbf{X}_{21}\left(\mathbf{X}^{11}\right)^{-1}\left(\mathbf{X}_{21}\right)^{\mathrm{H}}\right| \quad (64)$$

we get the final expression for the subchannel SNR MGF in case of transmit correlation. ■

D. Channel with receive correlation only

Before we analyze the most general case with both receive and transmit correlation in the next section, we first present a lemma for the case of receive correlation only, which is the starting point for later derivations.

Lemma 1: In case of receive correlation only, the subchannel SNR MGF has the integral representation

$$M_{\mathrm{RX}}(s) = \frac{1}{|s\cdot\gamma\cdot\mathbf{O}+\mathbf{I}_R|}\cdot\int\frac{\left|\mathbf{X}^{\mathrm{H}}\mathbf{O}\mathbf{X}\right|}{\left|\mathbf{X}^{\mathrm{H}}\mathbf{O}\left(s\cdot\gamma\cdot\mathbf{O}+\mathbf{I}_R\right)^{-1}\mathbf{X}\right|}\cdot\mathrm{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right)\ \mathrm{D}_c\mathbf{X}. \tag{65}$$

The integral can be interpreted as the expected value of a ratio of random determinants of complex Gaussian matrix quadratic forms. Obviously the subchannel SNR and its MGF, respectively, depends only on the eigenvalues of the receive correlation matrix.

Proof: If there is exclusively receive correlation present, the MIMO channel PDF is from (25) with the help of (43) given by

$$p_{\mathbf{K},RX}(\mathbf{K}) = \frac{1}{\pi^{RL}\cdot|\mathbf{R}_{\mathrm{RX}}|^L}\cdot\mathrm{etr}\left(-\left(\mathbf{k}_1^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\mathbf{k}_1+\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\tilde{\mathbf{K}}\right)\right). \tag{66}$$

It is independent of the subchannel index. Using relation (59) we get for the subchannel SNR MGF with the abbreviation $\boldsymbol{\Upsilon} = s\cdot\gamma\cdot\mathbf{I}_R+\mathbf{R}_{\mathrm{RX}}^{-1}$

$$M_{\mathrm{RX}}(s) = \frac{1}{(s\cdot\gamma)^{L-1}\cdot|\mathbf{R}_{\mathrm{RX}}|^L}\cdot\int\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|\cdot\mathrm{etr}\left(-\left(\mathbf{k}_1^{\mathrm{H}}\boldsymbol{\Upsilon}\mathbf{k}_1+\mathbf{k}_1^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}+\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{k}_1\right)\right)\cdot \tag{67}$$

$$\mathrm{etr}\left(-\left(\frac{1}{s\cdot\gamma}\cdot\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\mathbf{x}+\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\tilde{\mathbf{K}}\right)\right)\ \mathrm{D}_c\mathbf{k}_1\ \mathrm{D}_c\tilde{\mathbf{K}}\ \mathrm{D}_c\mathbf{x} \tag{68}$$

We first carry out the integral with respect to $\mathbf{k}_1$ and get

$$M_{\mathrm{RX}}(s) = \frac{1}{(s\cdot\gamma)^{L-1}\cdot|\mathbf{R}_{\mathrm{RX}}|^L\cdot|\boldsymbol{\Upsilon}|}\cdot\int\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|\cdot \tag{69}$$

$$\mathrm{etr}\left(-\left(\mathbf{x}^{\mathrm{H}}\tilde{\mathbf{K}}^{\mathrm{H}}\left(-\boldsymbol{\Upsilon}^{-1}+\frac{1}{s\cdot\gamma}\cdot\mathbf{I}_R\right)\tilde{\mathbf{K}}\mathbf{x}+\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\tilde{\mathbf{K}}\right)\right)\ \mathrm{D}_c\tilde{\mathbf{K}}\ \mathrm{D}_c\mathbf{x}$$

After integration with respect to $\mathbf{x}$ we find with the abbreviation $\boldsymbol{\Phi} = s\cdot\gamma\cdot\mathbf{R}_{\mathrm{RX}}+\mathbf{I}_R$

$$M_{\mathrm{RX}}(s) = \frac{1}{(s\cdot\gamma)^{L-1}\cdot|\mathbf{R}_{\mathrm{RX}}|^{L-1}\cdot|\mathbf{\Phi}|}\cdot\int\frac{\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|}{\left|\tilde{\mathbf{K}}^{\mathrm{H}}\left(-\mathbf{\Upsilon}^{-1}+\frac{1}{s\cdot\gamma}\cdot\mathbf{I}_R\right)\tilde{\mathbf{K}}\right|}\cdot \mathrm{etr}\left(-\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\tilde{\mathbf{K}}\right)\ \mathrm{D}_c\tilde{\mathbf{K}} \tag{70}$$

By using the matrix inversion lemma for square matrix $\mathbf{A}$ [24]

$$\mathbf{I}-\left(\mathbf{I}+\mathbf{A}^{-1}\right)^{-1}=(\mathbf{I}+\mathbf{A})^{-1} \tag{71}$$

we find

$$M_{\mathrm{RX}}(s) = \frac{1}{|\mathbf{\Phi}|}\cdot\int\frac{\left|\tilde{\mathbf{K}}^{\mathrm{H}}\tilde{\mathbf{K}}\right|}{\left|\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{\Phi}^{-1}\tilde{\mathbf{K}}\right|}\cdot \mathrm{etr}\left(-\tilde{\mathbf{K}}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{-1}\tilde{\mathbf{K}}\right)\cdot\frac{1}{|\mathbf{R}_{\mathrm{RX}}|^{L-1}}\ \mathrm{D}_c\tilde{\mathbf{K}}. \tag{72}$$

Now making the matrix variate transformation $\mathbf{R}_{\mathrm{RX}}^{-1/2}\tilde{\mathbf{K}}\to\mathbf{X}$ with Jacobian $J\left(\mathbf{R}_{\mathrm{RX}}^{-1/2}\tilde{\mathbf{K}}\to\mathbf{X}\right)=|\mathbf{R}_{\mathrm{RX}}|^{L-1}$ we get

$$M_{\mathrm{RX}}(s) = \frac{1}{|\mathbf{\Phi}|}\cdot\int\frac{\left|\mathbf{X}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}\mathbf{X}\right|}{\left|\mathbf{X}^{\mathrm{H}}\mathbf{R}_{\mathrm{RX}}^{1/2}(\mathbf{\Phi})^{-1}\mathbf{R}_{\mathrm{RX}}^{1/2}\mathbf{X}\right|}\cdot\mathrm{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right)\ \mathrm{D}_c\mathbf{X}. \tag{73}$$

By introducing the eigenvalue decomposition of the receive correlation matrix, we obtain the lemma. We note that the Jacobian $J(\mathbf{X}\cdot\mathbf{V}\to\mathbf{X})=1$ for unitary matrix $\mathbf{V}$. ■

Obviously, the MGF only depends on the eigenvalues of the receive correlation matrix and is independent of the particular eigenvectors. Before we continue with the calculation of the MGF, we further generalize the underlying channel model.

E. Transmit and receive correlation

Straightforward considerations lead to the following theorem.

Theorem 3: In case of Rayleigh fading with transmit and receive correlation we get similar to (65) a matrix variate integral expression for the subchannel SNR MGF

$$M_k(s) = \frac{1}{|s\cdot\tilde{\gamma}_k\cdot\mathbf{O}+\mathbf{I}_R|}\cdot\int\frac{\left|\mathbf{X}^{\mathrm{H}}\mathbf{O}\mathbf{X}\right|}{\left|\mathbf{X}^{\mathrm{H}}\mathbf{O}(s\cdot\tilde{\gamma}_k\cdot\mathbf{O}+\mathbf{I}_R)^{-1}\mathbf{X}\right|}\cdot\mathrm{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right)\ \mathrm{D}_c\mathbf{X}, \tag{74}$$

where $\left(c^{11},\ldots,c^{LL}\right)^T = \mathrm{diag}\left(\mathbf{C}^{-1}\right)$. The integral expression in (74) has again an interesting interpretation as the expected value of a ratio of random determinants in generalized matrix quadratic forms.

Proof: The theorem follows from a combination of the results for the transmit correlated only and receive correlated only cases above. Note that transmit correlation just leads to a scaling of the effective SNR according to (56). ∎

We now give a representation of the MGF in terms of a scalar integral only. It appears that there are no comparable results available in literature on the expected value of ratios of random determinants of complex Gaussian matrix quadratic forms. However, the formulas given in this paper cover the well known vector variate case [29] [30] [31] [32] [33] [34].

Theorem 4: The subchannel SNR MGF has the following single scalar integral representation (with matrix notation)

$$M_k(s) = \sum_{\hat{\alpha}_{L-1}} |\mathbf{O}|_{\hat{\alpha}_{L-1}}^{\hat{\alpha}_{L-1}} \cdot \int_0^\infty \frac{\mathrm{tr}\left(\mathbf{U}_{1,\alpha}\mathbf{U}_{2,\alpha}^{-1}\right)}{|\mathbf{U}_2|} \cdot t^{L-2}\, \mathrm{d}t, \tag{75}$$

where the sum is over all index subsets of $\{1,2,\ldots,R\}$ of cardinality $L-1$. For brevity, we have introduced

$$\begin{aligned}
\mathbf{U}_{1,\alpha} &= \mathbf{I}_{L-1} + s\cdot\frac{\gamma}{c^{kk}}\cdot\mathbf{O}_\alpha \qquad (76)\\
\mathbf{U}_2 &= \mathbf{I}_{L-1} + t\cdot\mathbf{O} + s\cdot\frac{\gamma}{c^{kk}}\cdot\mathbf{O}\\
\mathbf{U}_{2,\alpha} &= \mathbf{I}_{L-1} + t\cdot\mathbf{O}_\alpha + s\cdot\frac{\gamma}{c^{kk}}\cdot\mathbf{O}_\alpha
\end{aligned}$$

with $\mathbf{O}_\alpha = \{\mathbf{O}\}_{\hat{\alpha}_{L-1}}^{\hat{\alpha}_{L-1}}$. We get the scalar representation

$$M_k(s) = \sum_{\hat{\alpha}_{L-1}} |\mathbf{O}|_{\hat{\alpha}_{L-1}}^{\hat{\alpha}_{L-1}} \cdot \sum_{\alpha_l\in\hat{\alpha}_{L-1}} \cdot \int_0^\infty \frac{u_{1,\alpha_l}}{\left[\prod_{j=1}^R u_{2,j}\right]\cdot u_{2,\alpha_l}} t^{L-2}\, \mathrm{d}t, \tag{77}$$

where we have introduced

$$u_{1,\alpha_l} = 1+s\cdot\tilde{o}_{\alpha_l} \tag{78}$$

$$u_{2,l} = 1+t\cdot o_l+s\cdot\tilde{o}_l \tag{79}$$

$$u_{2,\alpha_l} = 1+t\cdot o_{\alpha_l}+s\cdot\tilde{o}_{\alpha_l}. \tag{80}$$

and $\tilde{o}_l = \tilde{\gamma}_k \cdot o_l$ according to (30).

Proof: By using Theorem 11 in Appendix III we can derive (75) after simple manipulations and (77) directly follows for diagonal matrices. ■

The complexity of the MGF expression can be reduced significantly. By deploying certain elementary symmetric functions, the following theorem can be derived. It is the starting point for later moment, SER, and MMI calculations.

Theorem 5: The MGF of the subchannel SNR in the presence if transmit and receive correlation has the concise scalar integral representation

$$M_k(s) = 1-s\cdot(L-1)\cdot\sum_k {o_k}^{R-1}\cdot\mathrm{tr}_{L-1}^{(k)}(\mathbf{O})\cdot K_{\mathbf{o}}(k)\cdot\int_0^\infty \frac{t^{L-2}}{s+\frac{1}{\tilde{o}_k}+\frac{1}{\tilde{\gamma}_k}\cdot t}\,\mathrm{d}t. \tag{81}$$

Carrying out the integral yields the closed-form solution

$$M_k(s) = 1+(-1)^L\cdot\tilde{\gamma}_k^{L-1}\cdot s\cdot(L-1)\cdot\sum_l \zeta_l, \tag{82}$$

where we have introduced the sum terms

$$\zeta_l = {o_l}^{R-1}\cdot\mathrm{tr}_{L-1}^{(l)}(\mathbf{O})\cdot K_{\mathbf{o}}(l)\cdot\left(s+\frac{1}{\tilde{o}_l}\right)^{L-2}\cdot\log\left(s+\frac{1}{\tilde{o}_l}\right) \tag{83}$$

for brevity and $\tilde{o}_l$ according to (30).

Proof: See Appendix IV. ■

F. Moments

Based on the closed form MGF expressions in Theorem 5, arbitrary moments of the subchannel SNR can be calculated.

Theorem 6: Let ν be the order of the moments. For the case $L > \nu$ the moments of the subchannel SNR are given by

$$m_k(\nu) = (-1)^{L-\nu-1} \cdot \tilde{\gamma}_k^{\nu} \cdot \frac{\nu \cdot \Gamma(L)}{\Gamma(L-\nu)} \cdot \sum_l o_l{}^{R-L+\nu} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l) \cdot \log \frac{1}{\tilde{o}_l}. \tag{84}$$

For the case $L \leq \nu$ the moments have the representation

$$m_k(\nu) = \tilde{\gamma}_k^{\nu} \cdot \nu \cdot \Gamma(L) \cdot \Gamma(\nu - L + 1) \cdot \sum_{j=1}^{\chi} (-1)^{j+1} \cdot \mathrm{h}_{\nu-L+1-j}(\mathbf{O}) \cdot \mathrm{tr}_{L-1+j}(\mathbf{O}) \tag{85}$$

with $\chi = \min(R-L+1, \nu-L+1)$.

Proof: If we want to calculate the νth moment, we get from the MGF by exchanging the sequence of differentiation and integration (this can be justified by Lebesgue's dominated convergence theorem; details are omitted here for brevity)

$$m_k(\nu) = (-1)^{\nu+1} \cdot (L-1) \cdot \sum_l o_l{}^{R-1} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l) \cdot \int_0^{\infty} \frac{\partial^{\nu}}{\partial s^{\nu}} \frac{t^{L-2} \cdot s}{s + \frac{1}{\tilde{o}_l} + \frac{1}{\tilde{\gamma}_k} \cdot t} \bigg|_{s=0} \mathrm{d}t. \tag{86}$$

Making use of the fact that

$$\frac{\partial^{\nu}}{\partial x^{\nu}} \frac{x}{x+a} \bigg|_{x=0} = (-1)^{\nu+1} \cdot \Gamma(\nu+1) \cdot \frac{1}{a^{\nu}} \tag{87}$$

we arrive at

$$m_k(\nu) = \Gamma(\nu+1) \cdot (L-1) \cdot \sum_l o_l{}^{R-1} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l) \cdot \int_0^{\infty} \frac{t^{L-2}}{\left(\frac{1}{\tilde{o}_l} + \frac{1}{\tilde{\gamma}_k} \cdot t\right)^{\nu}} \mathrm{d}t. \tag{88}$$

Integration by parts yields the following formula for integer $m > 0, n > 1$ and constants a, b

$$\int_0^{\infty} \frac{x^m}{(a+bx)^n} \mathrm{d}x = -\frac{x^m}{n-1} \cdot \frac{1}{b \cdot (a+bx)^{n-1}} \bigg|_0^{\infty} + \frac{1}{b} \frac{m}{n-1} \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{n-1}} \mathrm{d}x. \tag{89}$$

After application of (89) to (88) it can be readily seen by virtue of Lemma 5 in Appendix V that the first term resulting from (89) vanishes at the integration boundaries. Therefore we can find after iteratively applying (89) the following simplified integral formula for the case $L > \nu$ after simple modifications

$$m_k(\nu) \quad = \quad \tilde{\gamma}_k^{\nu-1} \cdot \frac{\nu \cdot \Gamma(L)}{\Gamma(L-\nu)} \cdot \sum_l {o_l}^{R-1} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l) \cdot \int_0^\infty \frac{t^{L-\nu-1}}{\frac{1}{\tilde{o}_l} + \frac{1}{\tilde{\gamma}_k} \cdot t} \,\mathrm{d}t. \tag{90}$$

Using (187) in Appendix IV for reformulating the fraction under the integral as a power series we find with the help of Lemma 5 in Appendix V

$$m_k(\nu) \quad = \quad (-1)^{L-\nu-1} \cdot \tilde{\gamma}_k^{L-1} \cdot \frac{\nu \cdot \Gamma(L)}{\Gamma(L-\nu)} \cdot \sum_k {o_k}^{R-1} \cdot \mathrm{tr}_{L-1}^{(k)}(\mathbf{O}) \cdot K_{\mathbf{o}}(k) \cdot \tilde{o}_k^{1+\nu-L} \cdot \log \frac{1}{\tilde{o}_k}. \tag{91}$$

This proves the first part of the theorem.

For the case $L \leq \nu$ we can apply the following integration formula valid for integer $m < n+1$ and constants a, b

$$\int_0^\infty \frac{x^m}{(a+bx)^n} \,\mathrm{d}x = \frac{\Gamma(n-m-1) \cdot \Gamma(m+1)}{a^{n-m-1} \cdot b^{m+1} \cdot \Gamma(n)} \tag{92}$$

and find for the moments

$$m_k(\nu) \quad = \quad \tilde{\gamma}_k^{\nu} \cdot \nu \cdot \Gamma(L) \cdot \Gamma(\nu - L + 1) \cdot \sum_l {o_l}^{R-L+\nu} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l). \tag{93}$$

This yields with the help of Lemma 6 in Appendix V the second part of the theorem.

■

V. PDF AND CDF OF SUBCHANNEL SNR

Starting with the MGF expression in Theorem 5, the subchannel SNR CDF and PDF can be calculated by inverse Laplace transforms.

Theorem 7: The CDF of the subchannel SNR is given by

$$q_k(\gamma_{\mathrm{SC},k}) \quad = \quad 1 - \Gamma(L) \cdot \sum_l \mu_{\mathbf{o},l} \cdot \left(\frac{\tilde{\gamma}_k}{\gamma_{\mathrm{SC},k}}\right)^{L-1} \cdot \exp\left(-\frac{1}{\tilde{o}_l} \gamma_{\mathrm{SC},k}\right) \tag{94}$$

with

$$\mu_{\mathbf{o},l} = {o_l}^{R-1} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l). \tag{95}$$

By expanding the exponential term in a power series in $\gamma_{\mathrm{SC},k}$, it can be shown by Lemma 5 in the appendix that $q(0)=0$. The corresponding subchannel SNR PDF has the representation

$$p_k(\gamma_{\mathrm{SC},k}) = \Gamma(L)\cdot\sum_l \mu_{\mathbf{o},l}\cdot\left(\frac{\tilde{\gamma}_k}{\gamma_{\mathrm{SC},k}}\right)^{L-1}\cdot\exp\left(-\frac{1}{\tilde{o}_l}\gamma_{\mathrm{SC},k}\right)\cdot\left(\frac{1}{\tilde{o}_l}+\frac{L-1}{\gamma_{\mathrm{SC},k}}\right). \tag{96}$$

Again, by an expansion of the exponential term it can be shown that for $R>L$ it is $p_k(0)=0$.

For completeness we note that in case of a transmit correlated or uncorrelated MIMO channel the PDF of the subchannel SNR is a well-known Gamma PDF with $\mathscr{D}$ degrees of freedom

$$p_{TX,k}(\gamma_{\mathrm{SC},k}) = \frac{1}{\Gamma(\mathscr{D})\cdot\tilde{\gamma}_k}\cdot\left(\frac{\gamma_{\mathrm{SC},k}}{\tilde{\gamma}_k}\right)^{\mathscr{D}-1}\cdot\exp\left(-\frac{\gamma_{\mathrm{SC},k}}{\tilde{\gamma}_k}\right). \tag{97}$$

Accordingly, we obtain for the CDF

$$q_{TX,k}(\gamma_{\mathrm{SC},k}) = 1-\exp\left(\frac{\gamma_{\mathrm{SC},k}}{\tilde{\gamma}_k}\right)\cdot\sum_{j=0}^{\mathscr{D}-1}\frac{1}{\Gamma(j+1)}\cdot\left(\frac{\gamma_{\mathrm{SC},k}}{\tilde{\gamma}_k}\right)^j. \tag{98}$$

Proof: The CDF $q_k(\gamma_{\mathrm{SC},k})=\int_0^{\gamma_{\mathrm{SC},k}} p(t)\mathrm{d}t$ has from Theorem 5 the Laplace transform

$$Q_k(s)=\frac{M_k(s)}{s} = \frac{1}{s}-(L-1)\cdot\sum_l \mu_{\mathbf{o},l}\cdot\int_0^\infty \frac{t^{L-2}}{s+\frac{1}{\tilde{o}_l}+\frac{1}{\tilde{\gamma}_k}\cdot t}\,\mathrm{d}t. \tag{99}$$

We can now make use of the Laplace transform pairs

$$\begin{aligned}\frac{1}{s+a} &\leftarrow e^{-ax} \\ \frac{1}{(s+a)^2} &\leftarrow xe^{-ax}.\end{aligned} \tag{100}$$

With an inverse Laplace transform we obtain

$$q_k(\gamma_{\mathrm{SC},k}) = 1-(L-1)\cdot\sum_l \mu_{\mathbf{o},l}\cdot\exp\left(-\frac{1}{\tilde{o}_l}\gamma_{\mathrm{SC},k}\right)\cdot\int_0^\infty t^{L-2}\cdot\exp\left(-\frac{1}{\tilde{\gamma}_k}\cdot\gamma_{\mathrm{SC},k}\cdot t\right)\,\mathrm{d}t. \tag{101}$$

Carrying out the integral we find the first part of the theorem. Finally, by differentiating with respect to $\gamma_{\mathrm{SC},k}$ we obtain the PDF $p_k(\gamma_{\mathrm{SC},k})$. ■

VI. SER CALCULATION

The conditional symbol error rate (conditioned on the subchannel SNR) in the presence of Gaussian noise for square M-QAM constellations is given by [19]

$$P_{s,k,cond} = b \cdot \left[Q\left(\sqrt{2c \cdot \gamma_{SC,k}}\right) - \frac{b}{4} \cdot Q^2\left(\sqrt{2c \cdot \gamma_{SC,k}}\right) \right] \tag{102}$$

with constants

$$\begin{aligned} b &= 4 \cdot \left(1 - \tfrac{1}{\sqrt{M}}\right) \\ c &= \tfrac{3}{2 \cdot (M-1)}. \end{aligned} \tag{103}$$

Based on a well known finite scalar integral representation of the Q function, we can use the closed form subchannel SNR MGF expressions for calculating exact SER formulas for square M-QAM constellations. An extension of the results to other QAM modulations is straightforward.

Theorem 8: The average SER of subchannel k of a MIMO ZF receiver in correlated Rayleigh fading with receive correlation is given by

$$P_{s,k} = b \cdot \left[1 - b/4 - \tilde{\gamma}_k^{L-1} \cdot (-1)^{L-1} \cdot \sum_l \varsigma_l \cdot \left[\Lambda_{1,l} + \frac{b}{\pi} \cdot \left(\Lambda_{2,l} + \Lambda_{3,l} \right) \right] \right] \tag{104}$$

with the auxiliary terms

$$\begin{aligned} \Lambda_{1,l} &= \sqrt{c \cdot \left(\frac{1}{\tilde{o}_l} + c \right)} \\ \Lambda_{2,l} &= -\sqrt{c \cdot \left(\frac{1}{\tilde{o}_l} + c \right)} \cdot \arctan \sqrt{1 + \frac{1}{c \cdot \tilde{o}_l}}, \end{aligned} \tag{105}$$

$$\varsigma_l = {o_l}^{R-1} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l) \cdot \frac{2^{L-1} \cdot (L-1)!}{(2L-3)!!} \cdot \left(\frac{1}{\tilde{o}_l} + c \right)^{L-2} \tag{106}$$

and

$$\Lambda_{3,l} = c \left[\frac{1}{2} + \sum_{l=0}^{L-3} \frac{(2L-2l-5)!! \cdot \left(\frac{1}{\tilde{o}_l} + 2c \right)^{L-2-l}}{2^{L-1-l} \cdot (L-2-l)! \cdot \left(\frac{1}{\tilde{o}_l} + c \right)^{L-2-l}} \right] \cdot \log \left(2 + \frac{1}{c \cdot \tilde{o}_l} \right) \tag{107}$$

for readability. In case of transmit correlation only or uncorrelated fading, the SER is given by

$$P_{s,k} = b \cdot \left[\Psi_1 - b/4 \cdot \left[\frac{1}{4} - \frac{\mu_c}{\pi} \left(\Psi_2 - \Psi_3 \right) \right] \right] \tag{108}$$

with the following terms for brevity

$$\mu_c = \sqrt{\frac{c\tilde{\gamma}_k}{1 + c\tilde{\gamma}_k}}, \tag{109}$$

$$\Psi_1 = \left(\frac{1-\mu_c}{2} \right)^L \cdot \sum_{l=0}^{L-1} \binom{L-1+l}{l} \left(\frac{1+\mu_c}{2} \right)^l, \tag{110}$$

$$\Psi_2 = \left(\frac{\pi}{2} - \arctan \mu_c \right) \cdot \sum_{l=0}^{L-1} \frac{\binom{2l}{l}}{\left[4 \left(1 + c\tilde{\gamma}_k \right) \right]^l}, \tag{111}$$

$$\Psi_3 = \sin\left(\arctan \mu_c \right) \cdot \sum_{l=1}^{L-1} \sum_{i=1}^{l} \frac{T_{il}}{\left(1 + c\tilde{\gamma}_k \right)^l} \cdot \left[\cos\left(\arctan \mu_c \right) \right]^{2(l-i)+1}, \tag{112}$$

and finally

$$T_{il} = \frac{\binom{2l}{l}}{\binom{2(l-i)}{l-i} \cdot 4^i \cdot \left[2(l-i) + 1 \right]}. \tag{113}$$

Proof: Due to space limitations, in this paper we omit a proof of the first SER formula of this theorem. It will be presented in the second part of this paper. The SER expression in (108) is a well known result from [19]. ■

A. Calculation of Mean Mutual Information

The mean mutual information (MMI) of MIMO subchannel k in nat per channel use is given by

$$\overline{I_k} = E \left[\log \left(1 + \gamma_{\mathrm{SC},k} \right) \right] \tag{114}$$

The expected value in (114) can be calculated using the CDF expression in (94) in closed form.

Theorem 9: The MMI of subchannel k of a MIMO link with ZF receiver in correlated Rayleigh fading is given by

$$\bar{I_k} = (-1)^{L-1} \cdot \Gamma(L) \cdot \tilde{\gamma}_k^{L-1} \cdot \sum_l \mu_{\mathbf{o},l} \cdot \left[E_1\left(\frac{1}{\tilde{o}_l}\right) \cdot \exp(-\frac{1}{\tilde{o}_l}) - \sum_{m=1}^{L-1} \frac{1}{\Gamma(m) \cdot \tilde{o}_l^{m-1}} \right], \tag{115}$$

where E_1 is the exponential integral [35], $\tilde{\gamma}_k$ from (29), $\tilde{\mathbf{o}}$ from (30), and

$$\mu_{\mathbf{o},l} = {o_l}^{R-1} \cdot \mathrm{tr}_{L-1}^{(l)}(\mathbf{O}) \cdot K_{\mathbf{o}}(l). \tag{116}$$

Proof: A proof is omitted in this paper due to the space limitation. It will be presented in part II of this paper, where we also consider the MGF of mutual information. ■

VII. Numerical Results

In this section we study systems with white input signals of power E_s and additive white Gaussian noise with variance N_0 per receive antenna

$$\begin{aligned} \mathbf{R}_{\mathrm{ss}} &= E_s \cdot \mathbf{I}_T \\ \mathbf{R}_{\mathrm{nn}} &= N_0 \cdot \mathbf{I}_R. \end{aligned} \tag{117}$$

Furthermore, due to their simple structure, in the following we consider exponential correlation matrices [36] at the transmitter and the receiver with

$$\begin{aligned} \mathbf{R}_{\mathrm{RX}} &= \left[r_{RX}^{|i-j|} \right] \\ \mathbf{R}_{\mathrm{TX}} &= \left[r_{TX}^{|i-j|} \right], \end{aligned} \tag{118}$$

where i and j are the row and column indices, respectively. The correlation coefficient at the receiver (transmitter) r_{RX} (r_{TX}) ranges from 0 to 1 and models the correlation between two neighboring receive (transmit) antennas. With the given channel model, correlation between two antenna elements decreases exponentially with their distance. Finally, the SNR in dB is defined by

$$\gamma_{dB} \equiv 10 \cdot \log_{10} \frac{\rho \cdot E_s}{N_0} = 10 \cdot \log_{10}(\rho \cdot \gamma) \quad [\text{dB}], \tag{119}$$

where ρ is the transmit power constraint and we assume in the following numerical results $\rho = T$ in accordance with (117).

In Fig. 2 we have plotted the empirical PDF of a MIMO system with ZF receiver and $T = 4$ transmit antennas, $L = 4$ independent subchannels, and $R = 6$ receive antennas. For the given scenario, we assume a MIMO channel with receive correlation only with $r_{RX} = 0.9$ and $r_{TX} = 0$. It is demonstrated that there is an exact match with the analytical PDF given in Theorem 7.

For the same channel correlation properties, the CDF is plotted according to Theorem 7 in Fig. 3.

The influence of receive correlation on the PDF can be seen in Fig. 4. With increasing correlation, as expected the PDF gets more peaky and the maximum of the PDF moves closer to zero. A considerable change of the PDF can be observed when the receive correlation coefficient r_{RX} is increased from 0.7 to 0.9.

In Fig. 5 we have plotted SER curves for a system with 16 QAM modulation and varying receive correlation. Theoretical results according to the closed form SER expressions in Theorem 8 and numerical results of a Monte Carlo simulation perfectly match. Again, the negative effect of receive correlation, especially for values $r_{RX} > 0.7$ can be observed.

SER curves for a system with again 16 QAM modulation are depicted in Fig. 5. We show curves for an uncorrelated channel as well as a receive correlated channel with $r_{RX} = 0.7$, whereas we note that due to symmetry considerations all subchannels for these two scenarios have the same SER. On the other hand, if there is additionally transmit correlation present with $r_{TX} = 0.7$, again due to symmetry there are two different SER on the subchannels.

In Fig. 7 SER curves for two systems with $R = \{4, 8\}$ receive antennas are depicted. Curves are shown for weakly and strongly correlated receive antennas $r_{RX} = \{0.3, 0.9\}$. Obviously, the full diversity of the systems with $\mathscr{D} = \{1, 5\}$ is achieved, independently of the strength of the receive correlation, for higher SNR. However, receive correlation leads to a considerable shift of the SER curves.

In Fig. 8 analytical curves of the MMI according to Theorem 9 and Monte Carlo simulation results perfectly agree for different scenarios with correlation at the receive antenna array.

VIII. Conclusion

For the first time we have determined the exact probability distribution of the statistics of MIMO ZF receivers in correlated Rayleigh fading with transmit as well as receive correlation. We have derived a novel probability distribution function that can be expressed in terms of certain elementary symmetric functions of the eigenvalues of the receive correlation matrix. Based on the closed form probability expressions, which are valid for an arbitrary finite number of transmit antennas, we have calculated exact formulas for the symbol error rate of square QAM constellations and presented results on mean mutual information. A new mathematical approach based on complex Gaussian integrals has been introduced for the derivation of the statistics. The authors expect that this approach will find numerous applications in other fields of information theory, particularly in the analysis of linear MIMO receivers like minimum mean squared error (MMSE) receivers.

Appendix I
Complex Gaussian Integrals

Basic material on real vector variate Gaussian integrals can be found in [37] and [38]. The straightforward extension to the complex case is e.g. given in [18]. For complex $m\times 1$ column vectors $\mathbf{x},\mathbf{a},\mathbf{b}$ and real positive definete matrix $\mathbf{A}$ the basic complex Gaussian integral is given by

$$\int \exp\left(-\mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}+\mathbf{a}^{\mathrm{H}}\mathbf{x}+\mathbf{x}^{\mathrm{H}}\mathbf{b}\right)\ \mathrm{D}_c\mathbf{x}=\frac{1}{|\mathbf{A}|}\exp\left(\mathbf{a}^{\mathrm{H}}\mathbf{A}^{-1}\mathbf{b}\right). \tag{120}$$

It can furthermore be shown that

$$\int \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}\cdot\exp\left(-\mathbf{x}^{\mathrm{H}}\mathbf{B}\mathbf{x}\right)\ \mathrm{D}_c\mathbf{x}=\frac{1}{|\mathbf{B}|}\cdot\mathrm{tr}\left(\mathbf{A}\mathbf{B}^{-1}\right). \tag{121}$$

Due to its importance in the derivations in this paper, we emphasize that from (120) we obtain the following integral representation of an inverse determinant

$$\frac{1}{|\mathbf{C}|}=\int \exp\left(-\mathbf{x}^{\mathrm{H}}\mathbf{C}\mathbf{x}\right)\ \mathrm{D}_c\mathbf{x} \tag{122}$$

Furthermore, (120) can be reformulated as

$$\exp\left(\mathbf{a}^{\mathrm{H}}\mathbf{A}^{-1}\mathbf{b}\right) = |\mathbf{A}| \cdot \int \exp\left(-\mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x} + \mathbf{a}^{\mathrm{H}}\mathbf{x} + \mathbf{x}^{\mathrm{H}}\mathbf{b}\right) \; \mathrm{D}_c\mathbf{x}, \tag{123}$$

i.e. we can get rid of the inverse in the exponent via an integral representation. In the matrix variate case we get similar to (120) for $M \times N$ matrices $\mathbf{X}, \mathbf{A}, \mathbf{B}$, $M \times M$ matrix $\mathbf{M}$, and $N \times N$ matrix $\mathbf{N}$ (see e.g. [6])

$$\int \mathrm{etr}\left(-\mathbf{N}\mathbf{X}^{H}\mathbf{M}\mathbf{X} + \mathbf{A}^{H}\mathbf{X} + \mathbf{X}^{H}\mathbf{B}\right) \; \mathrm{D}_c\mathbf{X} = \frac{1}{|\mathbf{M} \otimes \mathbf{N}|} \mathrm{etr}\left(\mathbf{N}^{-1}\mathbf{A}^{H}\mathbf{M}^{-1}\mathbf{B}\right) \tag{124}$$

with the special case

$$\int \mathrm{etr}\left(-\mathbf{X}^{H}\mathbf{M}\mathbf{X}\right) \; \mathrm{D}_c\mathbf{X} = \frac{1}{|\mathbf{M}|^{N}}. \tag{125}$$

Appendix II

Matrix Variate Distributions and Related Integrals

We base our derivations on certain expected values of random determinants for establishing some important integral equalities. In this context, we derive an exact closed form solution of the expected value of a noncentral matrix quadratic form and the corresponding matrix variate integral. It appears that this result until now was not available in literature in this explicit form.

First, we note that a noncentrally distributed complex Gaussian matrix $\bar{\mathbf{G}}$ of dimension $m \times n$ with i.i.d. elements of unity variance and mean $\mathbf{C}$ has the PDF

$$p_{\bar{\mathbf{G}}}(\bar{\mathbf{G}}) = \frac{1}{\pi^{mn}} \cdot \mathrm{etr}\left(-\left(\bar{\mathbf{G}} - \mathbf{C}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}} - \mathbf{C}\right)\right). \tag{126}$$

It was conjectured in [39] and finally proven in [40] that

$$E_{\bar{\mathbf{G}}}\left[\left|\bar{\mathbf{G}}^{\mathrm{H}}\bar{\mathbf{G}}\right|\right] = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} + \sum_{i=0}^{n-1} \frac{\Gamma(m-i)}{\Gamma(m-n+1)} \cdot \mathrm{tr}_{i+1}\left(\mathbf{Q}\right) \tag{127}$$

where $\mathbf{Q} = \mathbf{C}\mathbf{C}^{\mathrm{H}}$ for brevity and $\bar{\mathbf{G}}^{\mathrm{H}}\bar{\mathbf{G}}$ has a so-called complex noncentral Wishart distribution. Now note that from (127) together with (126) we can derive the important integral identity for rank 1 matrix $\mathbf{C}$

$$\int \left|\bar{\mathbf{G}}^{\mathrm{H}}\bar{\mathbf{G}}\right| \cdot \mathrm{etr}\left(-\left(\bar{\mathbf{G}} + \mathbf{C}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}} + \mathbf{C}\right)\right) \; \mathrm{D}_c\bar{\mathbf{G}} = \frac{1}{\Gamma(m-n+1)} \cdot \left[\Gamma(m+1) + \Gamma(m) \cdot \mathrm{tr}\left(\mathbf{Q}\right)\right]. \tag{128}$$

In the central case we directly get from (128)

$$\int \left|\mathbf{G}^{\mathrm{H}}\mathbf{G}\right| \cdot \mathrm{etr}\left(-\mathbf{G}^{\mathrm{H}}\mathbf{G}\right)\right) \; \mathrm{D}_c\mathbf{G} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}. \tag{129}$$

We now generalize the result in (128). First, we note that the $n \times n$ matrix $\bar{\mathbf{G}}^{\mathrm{H}}\mathbf{M}\bar{\mathbf{G}}$ with $m \times m$ matrix $\mathbf{M}$ is a complex noncentrally distributed generalized random matrix quadratic form. In the following, we calculate the expected value of a special random determinant.

Theorem 10: Consider the expected value of the random determinant

$$g = E_{\bar{\mathbf{G}}}\left[\left|\bar{\mathbf{G}}^{\mathrm{H}}\mathbf{M}\bar{\mathbf{G}}\right|\right] = \int \left|\bar{\mathbf{G}}^{\mathrm{H}}\mathbf{M}\bar{\mathbf{G}}\right| \cdot \mathrm{etr}\left(-\left(\bar{\mathbf{G}}+\mathbf{C}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}}+\mathbf{C}\right)\right) \; \mathrm{D}_c\bar{\mathbf{G}}, \tag{130}$$

where $\bar{\mathbf{G}}$ is noncentrally complex Gaussian distributed. It can be calculated in closed form

$$g = \sum_{\hat{\alpha}_n} |\mathbf{M}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot \left[\Gamma(n+1) + \sum_{i=0}^{n-1} \Gamma(n-i) \cdot \mathrm{tr}_{i+1}\left(\mathbf{C}_{1,\alpha}\mathbf{C}_{1,\alpha}^{\mathrm{H}}\right)\right], \tag{131}$$

where the sum is over all $\binom{n}{m}$ subsets of cardinality n and the $n \times n$ matrix

$$\mathbf{C}_{1,\alpha} = \{\mathbf{C}\}_{1,\ldots,n}^{\hat{\alpha}_n}. \tag{132}$$

In case of a rank 1 matrix $\mathbf{C}$ the result simplifies to

$$g_1 = \Gamma(n) \cdot \sum_{\hat{\alpha}_n} |\mathbf{M}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot \left[n + \mathrm{tr}\left(\mathbf{C}_{1,\alpha}\mathbf{C}_{1,\alpha}^{\mathrm{H}}\right)\right]. \tag{133}$$

Proof: We first expand the determinant expression in the integral of (130). To this end, we can make use of the general formula for $k \times k$ matrix $\mathbf{K} = \mathbf{C} \cdot \mathbf{D} \cdot \cdots \cdot \mathbf{R} \cdot \mathbf{S}$ (where matrices $\mathbf{C}, \mathbf{D}, \ldots, \mathbf{R}, \mathbf{S}$ are of compatible sizes)

$$|\mathbf{K}| = \sum_{\hat{\alpha}_k}\sum_{\hat{\beta}_k} \cdots \sum_{\hat{\delta}_k}\sum_{\hat{\sigma}_k} |\mathbf{C}|_{\hat{\alpha}_k}^{\{1,2,\ldots,k\}} \cdot |\mathbf{D}|_{\hat{\beta}_k}^{\hat{\alpha}_k} \cdot \cdots \cdot |\mathbf{R}|_{\hat{\sigma}_k}^{\hat{\delta}_k} \cdot |\mathbf{C}|_{\{1,2,\ldots,k\}}^{\hat{\sigma}_k}. \tag{134}$$

The sums in (134) are over all partitions $\hat{\alpha}_k, \hat{\beta}_k, \hat{\delta}_k, \hat{\sigma}_k$ of cardinality k. Direct application yields

$$\left|\bar{\mathbf{G}}^{\mathrm{H}}\mathbf{M}\bar{\mathbf{G}}\right| = \sum_{\hat{\alpha}_n} |\mathbf{M}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot \left|\bar{\mathbf{G}}\right|_{\hat{\alpha}_n}^{\{1,2,\ldots,n\}} \cdot \left|\bar{\mathbf{G}}\right|_{\{1,2,\ldots,n\}}^{\hat{\alpha}_n}. \tag{135}$$

Now we define a complementary index subset of cardinality $m-n$

$$\hat{\beta}_{m-n} = \{1,2,\ldots,m\} \backslash \hat{\alpha}_n \tag{136}$$

and for brevity we introduce the auxiliary matrices

$$\begin{aligned} \bar{\mathbf{G}}_{1,\alpha} &= \{\bar{\mathbf{G}}\}^{\hat{\alpha}_n}_{1,\ldots,n} \\ \bar{\mathbf{G}}_{2,\alpha} &= \{\bar{\mathbf{G}}\}^{\hat{\beta}_{m-n}}_{1,\ldots,n} \end{aligned} \tag{137}$$

and equivalently

$$\begin{aligned} \mathbf{C}_{1,\alpha} &= \{\mathbf{C}\}^{\hat{\alpha}_n}_{1,\ldots,n} \\ \mathbf{C}_{2,\alpha} &= \{\mathbf{C}\}^{\hat{\beta}_{m-n}}_{1,\ldots,n}. \end{aligned} \tag{138}$$

With the help of (135) and the partitionings (137)(138) we can rewrite (130) as

$$g = \sum_{\hat{\alpha}_n} |\mathbf{M}|^{\hat{\alpha}_n}_{\hat{\alpha}_n} \int \left|\bar{\mathbf{G}}^{\mathrm{H}}_{1,\alpha} \bar{\mathbf{G}}_{1,\alpha}\right| \cdot \operatorname{etr}\left(-\left(\bar{\mathbf{G}}+\mathbf{C}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}}+\mathbf{C}\right)\right) \ \mathrm{D}_c\bar{\mathbf{G}}. \tag{139}$$

We further focus on the integral in (139), which can be split into the product of two independent integrals

$$\begin{aligned} I_\alpha \;\; = \;\; & \int \left|\bar{\mathbf{G}}^{\mathrm{H}}_{1,\alpha} \bar{\mathbf{G}}_{1,\alpha}\right| \cdot \operatorname{etr}\left(-\left(\bar{\mathbf{G}}_{1,\alpha}+\mathbf{C}_{1,\alpha}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}}_{1,\alpha}+\mathbf{C}_{1,\alpha}\right)\right) \ \mathrm{D}_c\bar{\mathbf{G}}_{1,\alpha} \cdot \\ & \int \operatorname{etr}\left(-\left(\bar{\mathbf{G}}_{2,\alpha}+\mathbf{C}_{2,\alpha}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}}_{2,\alpha}+\mathbf{C}_{2,\alpha}\right)\right) \ \mathrm{D}_c\bar{\mathbf{G}}_{2,\alpha}. \end{aligned} \tag{140}$$

Using the matrix integral (127) we get

$$\begin{aligned} & \textstyle\int \left|\bar{\mathbf{G}}^{\mathrm{H}}_{1,\alpha} \bar{\mathbf{G}}_{1,\alpha}\right| \cdot \operatorname{etr}\left(-\left(\bar{\mathbf{G}}_{1,\alpha}+\mathbf{C}_{1,\alpha}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}}_{1,\alpha}+\mathbf{C}_{1,\alpha}\right)\right) \ \mathrm{D}_c\bar{\mathbf{G}}_{1,\alpha} = \\ & \Gamma(n+1) + \textstyle\sum_{i=0}^{n-1} \Gamma(n-i) \cdot \operatorname{tr}_{i+1}\left(\mathbf{C}_{1,\alpha}\mathbf{C}^{\mathrm{H}}_{1,\alpha}\right) \end{aligned} \tag{141}$$

and by straightforward considerations we find

$$\textstyle\int \operatorname{etr}\left(-\left(\bar{\mathbf{G}}_{2,\alpha}+\mathbf{C}_{2,\alpha}\right)^{\mathrm{H}}\left(\bar{\mathbf{G}}_{2,\alpha}+\mathbf{C}_{2,\alpha}\right)\right) \ \mathrm{D}_c\bar{\mathbf{G}}_{2,\alpha} = 1 \quad . \tag{142}$$

After combining the partial results we obtain the important theorem. ■

APPENDIX III
A RATIO OF RANDOM DETERMINANTS

In this Appendix, we explicitly calculate the expected value of a ratio of random determinants in complex generalized matrix quadratic forms. It appears that until now there were no results available in literature for this general matrix variate case, which also comprises the well-known vector variate case. In this paper, we give a scalar integral representation that is useful for the derivations of this paper. However, we note that the remaining integral can be calculated in closed form with the help of the residue theorem.

Theorem 11: Assume that $\mathbf{X}$ is a $m \times n$ complex Gaussian distributed matrix with i.i.d. elements and PDF

$$p_{\mathbf{X}}(\mathbf{X}) = \frac{1}{\pi^{mn}} \cdot \operatorname{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right). \tag{143}$$

The following expected value of random determinants

$$r = E_{\mathbf{X}}\left[\frac{|\mathbf{X}^{\mathrm{H}}\mathbf{C}\mathbf{X}|}{|\mathbf{X}^{\mathrm{H}}\mathbf{D}\mathbf{X}|}\right] \tag{144}$$

with respect to $\mathbf{X}$ with diagonal $m \times m$ matrices $\mathbf{C} = \operatorname{diag}(c_1, c_2, \ldots, c_m)$ and $\mathbf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_m)$ can be calculated by the single scalar integral expression

$$r = \sum_{\hat{\alpha}_n} |\mathbf{C}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot \int_0^\infty \frac{\operatorname{tr}\left(\left(\mathbf{I}_n + t \cdot \mathbf{D}_{1,\alpha}\right)^{-1}\right)}{|\mathbf{I}_m + t \cdot \mathbf{D}|} \cdot t^{n-1} \, \mathrm{d}t \tag{145}$$

with the auxiliary $n \times n$ matrix

$$\mathbf{D}_{1,\alpha} = \{\mathbf{D}\}_{1,\ldots,n}^{\hat{\alpha}_n}. \tag{146}$$

Proof: We first express the expected value as an integral

$$r = \int \frac{|\mathbf{X}^{\mathrm{H}}\mathbf{C}\mathbf{X}|}{|\mathbf{X}^{\mathrm{H}}\mathbf{D}\mathbf{X}|} \cdot \operatorname{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right) \, \mathrm{D}_c\mathbf{X}. \tag{147}$$

For carrying out the integral, we can use (122) and rewrite with $n \times 1$ vector $\mathbf{x}$

$$r = \int\int \left|\mathbf{X}^{\mathrm{H}}\mathbf{C}\mathbf{X}\right| \cdot \operatorname{etr}\left(-\mathbf{x}^{\mathrm{H}}\mathbf{X}^{\mathrm{H}}\mathbf{D}\mathbf{X}\mathbf{x}\right) \cdot \operatorname{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right)\ \mathrm{D}_c\mathbf{X}\ \mathrm{D}_c\mathbf{x}. \tag{148}$$

First, we focus on solving the integral

$$\tilde{r} = \int\int \left|\mathbf{X}^{\mathrm{H}}\mathbf{C}\mathbf{X}\right| \cdot \operatorname{etr}\left(\frac{1}{\kappa}\cdot\mathbf{x}^{\mathrm{H}}\mathbf{X}^{\mathrm{H}}\mathbf{D}\mathbf{X}\mathbf{x}\right) \cdot \operatorname{etr}\left(-\mathbf{X}^{\mathrm{H}}\mathbf{X}\right)\ \mathrm{D}_c\mathbf{X}\ \mathrm{D}_c\mathbf{x}, \tag{149}$$

where we have introduced a variable κ that is chosen such that the integrals that appear in the following derivations are formally correct and convergent. In the final result, it can be shown that the solution is valid for all κ and we let $\kappa = -1$ such that we can establish a solution for the problem in (148). In (149) we can introduce another integral expression with $m \times 1$ vector $\mathbf{y}$ according to (123) in Appendix I

$$\begin{aligned}\operatorname{etr}\left(\frac{1}{\kappa}\cdot\mathbf{x}^{\mathrm{H}}\mathbf{X}^{\mathrm{H}}\mathbf{D}\mathbf{X}\mathbf{x}\right) &= \frac{1}{\kappa^m}\cdot\int \operatorname{etr}\left(-\left(\kappa\cdot\mathbf{y}^{\mathrm{H}}\mathbf{y}+\mathbf{x}^{\mathrm{H}}\mathbf{X}^{\mathrm{H}}\mathbf{D}^{1/2}\mathbf{y}+\mathbf{y}^{\mathrm{H}}\mathbf{D}^{1/2}\mathbf{X}\mathbf{x}\right)\right)\ \mathrm{D}_c\mathbf{y} \quad (150)\\ &= \frac{1}{\kappa^m}\cdot\int \operatorname{etr}\left(-\kappa\cdot\mathbf{y}^{\mathrm{H}}\mathbf{y}\right)\cdot\operatorname{etr}\left(-\left(\mathbf{U}^{\mathrm{H}}\mathbf{X}+\mathbf{X}^{\mathrm{H}}\mathbf{U}\right)\right)\ \mathrm{D}_c\mathbf{y}\end{aligned}$$

with the auxiliary $m \times n$ matrix $\mathbf{U}$ for brevity

$$\mathbf{U} = \mathbf{D}^{1/2}\mathbf{y}\mathbf{x}^{\mathrm{H}}. \tag{151}$$

From (149) we obtain after straighforward manipulations and completing the square in the exponent

$$\tilde{r} = \frac{1}{\kappa^m}\cdot\int\int\int I_x\cdot\operatorname{etr}\left(-\kappa\cdot\mathbf{y}^{\mathrm{H}}\mathbf{y}\right)\cdot\operatorname{etr}\left(\mathbf{U}^{\mathrm{H}}\mathbf{U}\right)\ \mathrm{D}_c\mathbf{X}\ \mathrm{D}_c\mathbf{x}\ \mathrm{D}_c\mathbf{y} \tag{152}$$

with the auxiliary term for brevity

$$I_x = \left|\mathbf{X}^{\mathrm{H}}\mathbf{C}\mathbf{X}\right|\cdot\operatorname{etr}\left(-\left(\mathbf{X}+\mathbf{U}\right)^{\mathrm{H}}\left(\mathbf{X}+\mathbf{U}\right)\right). \tag{153}$$

The integral with respect to $\mathbf{X}$ can directly be solved (note again that auxiliary matrix $\mathbf{U}$ is of rank 1) via integral identity (133)

$$\tilde{r} \quad = \quad \frac{1}{\kappa^m} \cdot \Gamma(n) \cdot \sum_{\hat{\alpha}_n} |\mathbf{C}|_{\hat{\alpha}_n}^{\hat{\alpha}_n} \cdot \int\int I_{xy} \ \mathrm{D}_c\mathbf{x} \ \mathrm{D}_c\mathbf{y} \tag{154}$$

with

$$I_{xy} = \left[n + \operatorname{tr}\left(\mathbf{U}_{1,\alpha}\mathbf{U}_{1,\alpha}^{\mathrm{H}}\right)\right] \cdot \operatorname{etr}\left(-\kappa \cdot \mathbf{y}^{\mathrm{H}}\mathbf{y}\right) \cdot \operatorname{etr}\left(\mathbf{U}^{\mathrm{H}}\mathbf{U}\right). \tag{155}$$

After introducing the partitionings of the matrix $\mathbf{D}$

$$\begin{aligned} \mathbf{D}_{1,\alpha} &= \{\mathbf{D}\}_{\hat{\alpha}_n}^{\hat{\alpha}_n} \\ \mathbf{D}_{2,\alpha} &= \{\mathbf{D}\}_{\hat{\beta}_{m-n}}^{\hat{\beta}_{m-n}} \end{aligned} \tag{156}$$

and the vector $\mathbf{y}$

$$\begin{aligned} \mathbf{y}_{1,\alpha} &= \{\mathbf{y}\}_{\{1\}}^{\hat{\alpha}_n} \\ \mathbf{y}_{2,\alpha} &= \{\mathbf{y}\}_{\{1\}}^{\hat{\beta}_{m-n}} \end{aligned} \tag{157}$$

we get for the two parts of auxiliary matrix $\mathbf{U}$

$$\begin{aligned} \mathbf{U}_{1,\alpha} &= \mathbf{D}_{1,\alpha}^{1/2}\mathbf{y}_{1,\alpha}\mathbf{x}^{\mathrm{H}} \\ \mathbf{U}_{2,\alpha} &= \mathbf{D}_{2,\alpha}^{1/2}\mathbf{y}_{2,\alpha}\mathbf{x}^{\mathrm{H}} \quad . \end{aligned} \tag{158}$$

Using (158) in the integral expression of (154) we get

$$\begin{aligned} r_\alpha \quad = \quad & \int\int \left[n + \mathbf{y}_{1,\alpha}^{\mathrm{H}}\mathbf{D}_{1,\alpha}\mathbf{y}_{1,\alpha}\mathbf{x}^{\mathrm{H}}\mathbf{x}\right] \operatorname{etr}\left(-\mathbf{y}_{1,\alpha}^{\mathrm{H}}\left(\kappa \cdot \mathbf{I}_n - \mathbf{x}^{\mathrm{H}}\mathbf{x}\mathbf{D}_{1,\alpha}\right)\mathbf{y}_{1,\alpha}\right) \ \mathrm{D}_c\mathbf{y}_{1,\alpha} \cdot \\ & \int \operatorname{etr}\left(-\mathbf{y}_{2,\alpha}^{\mathrm{H}}\left(\kappa \cdot \mathbf{I}_{m-n} - \mathbf{x}^{\mathrm{H}}\mathbf{x}\mathbf{D}_{2,\alpha}\right)^{\mathrm{H}}\mathbf{y}_{2,\alpha}\right) \ \mathrm{D}_c\mathbf{y}_{2,\alpha} \ \mathrm{D}_c\mathbf{x}. \end{aligned} \tag{159}$$

We can make use of (120) and (121) for calculating the integrals and obtain

$$r_\alpha \quad = \quad \int \frac{1}{|\kappa \cdot \mathbf{I}_m - \mathbf{x}^{\mathrm{H}}\mathbf{x} \cdot \mathbf{D}|}\left[n + \operatorname{tr}\left(\mathbf{x}^{\mathrm{H}}\mathbf{x} \cdot \mathbf{D}_{1,\alpha}\left(\kappa \cdot \mathbf{I}_n - \mathbf{x}^{\mathrm{H}}\mathbf{x} \cdot \mathbf{D}_{1,\alpha}\right)^{-1}\right)\right] \ \mathrm{D}_c\mathbf{x}. \tag{160}$$

This can be further simplified via application of the general formula for $n \times 1$ vector $\mathbf{x}$

$$\int f(\mathbf{x}^{\mathrm{H}}\mathbf{x})\ \mathrm{D}_c\mathbf{x} = \frac{1}{\Gamma(n)} \int_0^\infty f(t)\cdot t^{n-1}\ \mathrm{d}t, \tag{161}$$

which can be derived via a variable transformation to polar coordinates. We obtain

$$r_\alpha = \frac{(-1)^m}{\Gamma(n)} \int_0^\infty \frac{1}{|-\kappa\cdot\mathbf{I}_m + t\cdot\mathbf{D}|}\mathrm{tr}\left(\mathbf{I}_n - t\cdot\mathbf{D}_{1,\alpha}\left(-\kappa\cdot\mathbf{I}_n + t\cdot\mathbf{D}_{1,\alpha}\right)^{-1}\right)\cdot t^{n-1}\ \mathrm{d}t, \tag{162}$$

which can be simplified with the matrix inversion lemma to

$$r_\alpha = \frac{(-1)^m}{\Gamma(n)} \int_0^\infty \frac{1}{|-\kappa\cdot\mathbf{I}_m + t\cdot\mathbf{D}|}\cdot\mathrm{tr}\left(\left(\mathbf{I}_n - \frac{t}{\kappa}\cdot\mathbf{D}_{1,\alpha}\right)^{-1}\right)\cdot t^{n-1}\ \mathrm{d}t. \tag{163}$$

As an important result, the expression in (163) becomes for $\kappa = -1$

$$r_\alpha|_{\kappa=-1} = \frac{(-1)^m}{\Gamma(n)} \int_0^\infty \frac{1}{|\mathbf{I}_m + t\cdot\mathbf{D}|}\cdot\mathrm{tr}\left(\left(\mathbf{I}_n + t\cdot\mathbf{D}_{1,\alpha}\right)^{-1}\right)\cdot t^{n-1}\ \mathrm{d}t. \tag{164}$$

The integral is convergent, as the integrand has no poles in the integration interval and behaves like $\frac{1}{t^{m-n+2}}$ for large t. Substituting (164) in (154) we arrive at the single scalar integral expression given in the theorem. ■

APPENDIX IV
PROOF OF THEOREM 5

With $q_k = -\frac{1+t\cdot o_k}{\tilde{o}_k} = -\frac{1}{\tilde{\gamma}_k}\cdot\left(\frac{1}{o_k}+t\right)$ we get from Theorem 4 the equivalent MGF representation

$$M_k(s) = \frac{1}{|\tilde{\gamma}_k\mathbf{O}|}\cdot\sum_{\hat{\alpha}_{L-1}} |\mathbf{O}|_{\hat{\alpha}_{L-1}}^{\hat{\alpha}_{L-1}}\cdot\sum_{\alpha_m\in\hat{\alpha}_{L-1}} \int_0^\infty \frac{s+\frac{1}{\tilde{o}_{\alpha_m}}}{\left[\prod_{l=1,l\neq\alpha_m}^{R}(s-q_l)\right]\cdot(s-q_{\alpha_m})^2}\cdot t^{L-2}\ \mathrm{d}t. \tag{165}$$

We can now decompose the integrand into partial fractions with respect to s

$$\frac{s+\frac{1}{\tilde{o}_{\alpha_m}}}{\left[\prod_{l=1,l\neq\alpha_m}^{R}(s-q_l)\right]\cdot(s-q_{\alpha_m})^2} t^{L-2} = \sum_{l=1,l\neq\alpha_m}^{R} X_l(\alpha_m) + Y_1(\alpha_m) + Y_2(\alpha_m). \tag{166}$$

With the short-hand notations

$$F_l = s + \frac{1}{\tilde{o}_l} + \frac{1}{\tilde{\gamma}_l} \cdot t \tag{167}$$

and

$$Z_{\mathbf{o}}(l) = \frac{1}{\prod_{n=1,n\neq l}^{R} \left(\frac{1}{o_l} - \frac{1}{o_n} \right)} \tag{168}$$

we get

$$\begin{aligned} X_{l\neq\alpha_m}(\alpha_m) &= (-\tilde{\gamma}_k)^{R-1} \cdot Z_{\mathbf{o}}(l) \cdot \frac{1}{\frac{1}{o_l} - \frac{1}{o_{\alpha_m}}} \cdot \frac{t^{L-1}}{F_l} + \\ &\quad (-\tilde{\gamma}_k)^{R-1} \cdot Z_{\mathbf{o}}(l) \cdot \frac{t^{L-2}}{F_l} \\ &= X_{l\neq\alpha_m,1}(\alpha_m) + X_{l\neq\alpha_m,2}(\alpha_m). \end{aligned} \tag{169}$$

$$\begin{aligned} Y_1(\alpha_m) &= (-\tilde{\gamma}_k)^{R-1} \cdot Z_{\mathbf{o}}(\alpha_m) \cdot \frac{t^{L-2}}{F_{\alpha_m}} + \\ &\quad (-\tilde{\gamma}_k)^{R} \cdot Z_{\mathbf{o}}(\alpha_m) \cdot \sum_{n=1,n\neq\alpha_m}^{R} \frac{1}{\frac{1}{o_{\alpha_m}} - \frac{1}{o_n}} \cdot \frac{t^{L-1}}{F_{\alpha_m}} \\ &= Y_{11}(\alpha_m) + Y_{12}(\alpha_m). \end{aligned} \tag{170}$$

$$Y_2(\alpha_m) = (-\tilde{\gamma}_k)^{R-2} \cdot Z_{\mathbf{o}}(\alpha_m) \cdot \frac{t^{L-1}}{(F_{\alpha_m})^2}. \tag{171}$$

Using integration by parts we obtain

$$\int_0^\infty Y_2(\alpha_m)\ \mathrm{d}t = (-\tilde{\gamma}_k)^{R-1} \cdot Z_{\mathbf{o}}(\alpha_m) \cdot \frac{t^{L-1}}{F_{\alpha_m}} \bigg|_0^\infty - \tag{172}$$

$$(L-1) \cdot \int_0^\infty (-\tilde{\gamma}_k)^{R-1} \cdot Z_{\mathbf{o}}(\alpha_m) \cdot \frac{t^{L-2}}{F_{\alpha_m}} \mathrm{d}t \tag{173}$$

$$= \tilde{Y}_{21}(\alpha_m) - (L-1) \cdot \int_0^\infty \tilde{Y}_{22}(\alpha_m)\,\mathrm{d}t.$$

In the original equation (165) the terms Y_{11} and $\tilde{Y}_{22}$ cancel and we find after some tedious algebra

$$M_k(s) = \frac{1}{\tilde{\gamma}_k} \cdot \sum_{\hat{\alpha}_{L-1}} |\mathbf{O}|^{\hat{\alpha}_{L-1}}_{\hat{\alpha}_{L-1}} \cdot \sum_{\alpha_m \in \hat{\alpha}_{L-1}} \left(U\left(\alpha_m\right) + I_{\alpha_m}\right). \tag{174}$$

The main terms in (174) are

$$I_{\alpha_m} = \int_0^\infty \left[\sum_{l=1, l\neq\alpha_m}^{R} V_l\left(\alpha_m\right) + W\left(\alpha_m\right) - (L-2)\cdot Q\left(\alpha_m\right) \right] \mathrm{d}t, \tag{175}$$

$$\begin{aligned} V_{l\neq\alpha_m}\left(\alpha_m\right) &= {o_l}^{R-1} \cdot o_{\alpha_m} \cdot \tilde{K}_{\mathbf{o}}\left(\alpha_m, l\right) \cdot \frac{K_{\mathbf{o}}\left(l\right)}{F_l} \cdot t^{L-1} + \\ &\quad {o_l}^{R-2} \cdot \frac{K_{\mathbf{o}}\left(l\right)}{F_l} \cdot t^{L-2}, \end{aligned} \tag{176}$$

$$W\left(\alpha_m\right) = {o_{\alpha_m}}^{R-1} \cdot \left[\sum_{n=1, n\neq\alpha_m}^{R} o_n \cdot \tilde{K}_{\mathbf{o}}\left(\alpha_m, n\right) \right] \cdot \frac{K_{\mathbf{o}}\left(\alpha_m\right)}{F_{\alpha_m}} \cdot t^{L-1}, \tag{177}$$

$$U\left(\alpha_m\right) = {o_{\alpha_m}}^{R-2} \cdot K_{\mathbf{o}}\left(\alpha_m\right) \cdot \left. \frac{t^{L-1}}{F_{\alpha_m}} \right|_{t=0}^{\infty}, \tag{178}$$

and finally

$$Q\left(\alpha_m\right) = {o_{\alpha_m}}^{R-2} \cdot K_{\mathbf{o}}\left(\alpha_m\right) \cdot \frac{t^{L-2}}{F_{\alpha_m}}. \tag{179}$$

For the reformulation we have used

$$\frac{1}{|\mathbf{O}|} \cdot Z_{\mathbf{o}}\left(l\right) = K_{\mathbf{o}}\left(l\right) \cdot o_l^{R-2} \cdot (-1)^{R-1}. \tag{180}$$

The MGF can be further simplified. First, we do a resummation

$$M_k(s) = \frac{1}{\tilde{\gamma}_k} \cdot \sum_m o_m \cdot \mathrm{tr}^{(m)}_{L-2}\left(\mathbf{O}\right) \left(U\left(m\right) + \int_0^\infty \sum_{l=1, l\neq m}^{R} V_l\left(m\right) + W\left(m\right) - (L-2)\cdot Q\left(m\right) \; \mathrm{d}t \right). \tag{181}$$

After a rearrangement of the terms, we obtain

$$\begin{aligned} M_k(s) \;=\; & \frac{1}{\tilde{\gamma}_k}\cdot\sum_m \mathrm{tr}_{L-2}^{(m)}(\mathbf{O})\cdot o_m^{R-1}\cdot K_{\mathbf{o}}(m)\cdot\left.\frac{t^{L-1}}{F_m}\right|_{t=0}^{\infty} \\ & + {o_m}^{R-1}\cdot\int_0^\infty\left[\sum_{l=1,l\neq m}^{R} o_l\cdot\mathrm{tr}_{L-2}^{(l)}(\mathbf{O})\cdot\frac{1}{o_m}\cdot\frac{K_{\mathbf{o}}(m)}{F_m}\cdot t^{L-2}\right. \\ & + \left(\mathrm{tr}_{L-2}^{(m)}(\mathbf{O})\cdot o_m\cdot\sum_{l=1,l\neq m}^{R} o_l\cdot\tilde{K}_{\mathbf{o}}(m,l)+\sum_{l=1,l\neq m}^{R} o_l^2\cdot\mathrm{tr}_{L-2}^{(l)}(\mathbf{O})\cdot\tilde{K}_{\mathbf{o}}(l,m)\right)\cdot\frac{K_{\mathbf{o}}(m)}{F_m}\cdot t^{L-1} \\ & \left. -(L-2)\cdot\mathrm{tr}_{L-2}^{(m)}(\mathbf{O})\cdot K_{\mathbf{o}}(m)\cdot\frac{t^{L-2}}{F_m}\;\mathrm{d}t\right]. \end{aligned} \tag{182}$$

A first simplification with the help of Lemma 8 in Appendix V yields

$$\begin{aligned} M_k(s) \;=\; & \frac{1}{\tilde{\gamma}_k}\cdot\sum_m \mathrm{tr}_{L-2}^{(m)}(\mathbf{O})\cdot o_m^{R-1}\cdot K_{\mathbf{o}}(m)\cdot\left.\frac{t^{L-1}}{F_m}\right|_{t=0}^{\infty} \\ & + {o_m}^{R-1}\cdot\int_0^\infty\left[\sum_{l=1,l\neq m}^{R} o_l\cdot\mathrm{tr}_{L-2}^{(l)}(\mathbf{O})\cdot\frac{1}{o_m}\cdot\frac{K_{\mathbf{o}}(m)}{F_m}\cdot t^{L-2}\right. \\ & + (L-1)\cdot\mathrm{tr}_{L-1}^{(m)}(\mathbf{O})\cdot\frac{K_{\mathbf{o}}(m)}{F_m}\cdot t^{L-1} \\ & \left. -(L-2)\cdot\mathrm{tr}_{L-2}^{(m)}(\mathbf{O})\cdot K_{\mathbf{o}}(m)\cdot\frac{t^{L-2}}{F_m}\;\mathrm{d}t\right]. \end{aligned} \tag{183}$$

Application of Lemma 4 in Appendix V yields the simplification

$$\begin{aligned} M_k(s) \;=\; & \frac{1}{\tilde{\gamma}_k}\cdot\sum_m \mathrm{tr}_{L-2}^{(m)}(\mathbf{O})\cdot o_m^{R-1}\cdot K_{\mathbf{o}}(m)\cdot\left.\frac{t^{L-1}}{F_m}\right|_{t=0}^{\infty} \\ & + {o_m}^{R-1}\cdot\int_0^\infty\left[(L-1)\cdot\mathrm{tr}_{L-1}^{(m)}(\mathbf{O})\cdot\frac{1}{o_m}\cdot\frac{K_{\mathbf{o}}(k)}{F_m}\cdot t^{L-2}\right. \\ & \left. + (L-1)\cdot\mathrm{tr}_{L-1}^{(m)}(\mathbf{O})\cdot\frac{K_{\mathbf{o}}(m)}{F_m}\cdot t^{L-1}\;\mathrm{d}t\right]. \end{aligned} \tag{184}$$

Then note that

$$\frac{1}{o_l\cdot F_l}+\frac{t}{F_l}=\tilde{\gamma}_k\cdot\left(1-\frac{s}{F_l}\right). \tag{185}$$

We thus find from (184)

$$M_k(s) = \frac{1}{\tilde{\gamma}_k} \cdot \sum_m \mathrm{tr}_{L-2}^{(m)}(\mathbf{O}) \cdot o_m^{R-1} \cdot K_{\mathbf{o}}(m) \cdot \left. \frac{t^{L-1}}{F_m} \right|_{t=0}^{\infty} - \tilde{\gamma}_k \cdot s \cdot {o_m}^{R-1} \cdot (L-1) \cdot \mathrm{tr}_{L-1}^{(m)}(\mathbf{O}) \cdot K_{\mathbf{o}}(m) \cdot \int_0^{\infty} \frac{t^{L-2}}{F_m} \, \mathrm{d}t. \tag{186}$$

We can make use of the formula

$$\frac{x^n}{a+bx} = \frac{(-1)^n \cdot a^{n-1}}{b^n} \cdot \left[\frac{1}{1+\frac{b}{a}x} - \sum_{i=0}^{n-1} (-1)^i \left(\frac{b}{a} x \right)^i \right] \tag{187}$$

for rewriting the term $\frac{t^{L-1}}{F_m}$.

Using Lemma 5 in Appendix V for simplifying the sum resulting from application of (187) we can finally prove the first part of the theorem.

APPENDIX V
ELEMENTARY SYMMETRIC FUNCTIONS

A powerful tool for deriving identities for elementary symmetric functions is the generating function (GF) approach. For the elementary symmetric functions (ESF) of the $m \times 1$ vector $\mathbf{x}$ it reads

$$E(\mathbf{x}, t) = \prod_{l=1}^{m} (1 + x_l \cdot t) = \sum_{l=0}^{m} \mathrm{tr}_l(\mathbf{x}) \cdot t^l. \tag{188}$$

We use (188) to derive a number of important ESF identities.

Lemma 2: For $0 \leq n \leq m-1$ and $1 \leq k \leq m$

$$\mathrm{tr}_{n+1}(\mathbf{x}) = \mathrm{tr}_{n+1}^{(k)}(\mathbf{x}) + x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}). \tag{189}$$

Proof: We can rewrite the GF as

$$E(\mathbf{x}, t) = \prod_{l=1, l \neq k}^{m} (1 + x_l \cdot t) + x_k \cdot t \cdot \prod_{l=1, l \neq k}^{m} (1 + x_l \cdot t). \tag{190}$$

Using (188) we obtain the equation

$$\sum_{l=0}^{m} \mathrm{tr}_l(\mathbf{x}) \cdot t^l = \sum_{l_1=0}^{m-1} \mathrm{tr}_{l_1}^{(k)}(\mathbf{x}) \cdot t^{l_1} + x_k \cdot t \cdot \sum_{l_2=0}^{m-1} \mathrm{tr}_{l_2}^{(k)}(\mathbf{x}) \cdot t^{l_2}. \tag{191}$$

By equating coefficients of like power in t we can establish the lemma. ■

Lemma 3: For $m\times 1$ vector $\mathbf{x}$ the following relation holds

$$\sum_{l=1}^{m} x_l \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) = (n+1)\cdot \mathrm{tr}_{n+1}(\mathbf{x}). \tag{192}$$

Proof: Differentiating the GF we can derive

$$\frac{\partial}{\partial t} E(\mathbf{x},t) = \sum_{k=1}^{m} x_k \cdot \prod_{l=1, l\neq k}^{m} (1+x_l\cdot t) = \sum_{r=1}^{m} r\cdot \mathrm{tr}_r(\mathbf{x})\cdot t^{r-1}. \tag{193}$$

By equating coefficients of like power in t we can establish the lemma. ■

Lemma 4: For $m\times 1$ vector $\mathbf{x}$ the following relation holds

$$\sum_{l=1}^{m} x_l \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) = (n+1)\cdot \left[\mathrm{tr}_{n+1}^{(k)}(\mathbf{x}) + x_k\cdot \mathrm{tr}_n^{(k)}(\mathbf{x})\right]. \tag{194}$$

Proof: The lemma directly follows from application of Lemma 2 and Lemma 3. ■

Lemma 5: For $k=0\ldots m-1$ we have

$$\sum_{l=1}^{m} x_l^k \cdot \mathrm{tr}_n^{(l)}(\mathbf{x})\cdot K_{\mathbf{x}}(l) = (-1)^n\cdot \delta(k-(m-n-1)). \tag{195}$$

For $m>k>m-n-1$ we therefore find the important special case

$$\sum_{l=1}^{m} x_l^k \cdot \mathrm{tr}_n^{(l)}(\mathbf{x})\cdot K_{\mathbf{x}}(l) = 0 \ \forall\ m>k>m-n-1. \tag{196}$$

Proof: We begin the proof with the expansion in partial fractions in Lemma 9 in Appendix VI for $0\le\mu\le m$

$$\begin{aligned}
\frac{(-1)^{\mu}\cdot t^{\mu}}{\prod_{i=1}^{m}(1+x_i\cdot t)} &= \sum_{l=1}^{m} K_{\mathbf{x}}(l)\cdot \frac{x_l^{m-\mu-1}}{1+x_l\cdot t} \\
&= \sum_{l=1}^{m} \frac{K_{\mathbf{x}}(l)\cdot x_l^{m-\mu-1}\cdot \prod_{j=1,j\neq\mu}^{m}\left(1+x_j\cdot t\right)}{\prod_{\nu=1}^{m}(1+x_\nu\cdot t)} \\
&= \frac{\sum_{l=1}^{m} K_{\mathbf{x}}(l)\cdot x_l^{m-\mu-1}\cdot \sum_{n=0}^{m-1}\mathrm{tr}_n^{(l)}(\mathbf{x})\cdot t^n}{\prod_{\nu=1}^{m}(1+x_\nu\cdot t)}
\end{aligned} \tag{197}$$

By comparing like powers of t we find

$$(-1)^{\mu}\cdot\delta(\mu-n) = \sum_{l=1}^{m} K_{\mathbf{x}}(l)\cdot x_l^{m-\mu-1}\cdot \mathrm{tr}_n^{(l)}(\mathbf{x}). \tag{198}$$

Then setting $k=m-\mu-1$ proves the lemma. ■

Lemma 6: For $k \geq m$ we have with $\tau=\min(m-n,k+1-m)$

$$\sum_{l=1}^{m} x_l^k \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) \cdot K_{\mathbf{x}}(l) = \sum_{j=0}^{\tau} (-1)^{j+1} \cdot \mathrm{h}_{k+1-m}(\mathbf{x}) \cdot \mathrm{tr}_{n+j}(\mathbf{x}). \tag{199}$$

Proof: We begin the proof with the expansion in partial fractions in Lemma 10 in Appendix VI

$$\begin{aligned} \frac{(-1)^{\mu}}{t^{\mu} \cdot \prod_{i=1}^{m}(1+x_i \cdot t)} &= \sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot \frac{x_l^{m+\mu-1}}{1+x_l \cdot t} + \sum_{j=1}^{\mu} (-1)^j \cdot \frac{\mathrm{h}_{\mu-j}(\mathbf{x})}{t^j} \\ &= \frac{\gamma_1+\gamma_2}{t^{\mu} \cdot \prod_{\nu=1}^{m}(1+x_{\nu} \cdot t)} \\ &= \frac{\delta_1+\delta_2}{t^{\mu} \cdot \prod_{\nu=1}^{m}(1+x_{\nu} \cdot t)} \end{aligned} \tag{200}$$

with the auxiliary terms

$$\begin{aligned} \gamma_1 &= \sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot x_l^{m+\mu-1} \cdot t^{\mu} \cdot \prod_{i=1, i\neq\mu}^{m} (1+x_i \cdot t) \\ \gamma_2 &= \sum_{j=1}^{\mu} (-1)^j \cdot \mathrm{h}_{\mu-j}(\mathbf{x}) \cdot t^{\mu-j} \cdot \prod_{n=1}^{m} (1+x_n \cdot t) \end{aligned} \tag{201}$$

and

$$\begin{aligned} \delta_1 &= \sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot x_l^{m+\mu-1} \cdot \sum_{i=1}^{m-1} \mathrm{tr}_i^{(l)}(\mathbf{x}) \cdot t^{i+\mu} \\ \delta_2 &= \sum_{j=1}^{\mu} (-1)^j \cdot \mathrm{h}_{\mu-j}(\mathbf{x}) \cdot \sum_{n=1}^{m} \mathrm{tr}_n(\mathbf{x}) \cdot t^{\mu+n-j} \end{aligned} \tag{202}$$

for brevity. By comparing both sides for $\mu \geq 1$ we find

$$\sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot x_l^{m+\mu-1} \cdot \sum_{i=1}^{m-1} \mathrm{tr}_i^{(l)}(\mathbf{x}) \cdot t^{i+\mu} + \sum_{j=1}^{\mu} (-1)^j \cdot \mathrm{h}_{\mu-j}(\mathbf{x}) \cdot \sum_{n=1}^{m} \mathrm{tr}_n(\mathbf{x}) \cdot t^{\mu+n-j} = 0. \tag{203}$$

Now we compare like powers of t on both sides. To this end, we consider (203) for fixed i

$$\sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot x_l^{m+\mu-1} \cdot \mathrm{tr}_i^{(l)}(\mathbf{x}) \cdot t^{i+\mu} = \sum_{j=1}^{\min(m-i,\mu)} (-1)^{j+1} \cdot \mathrm{h}_{\mu-j}(\mathbf{x}) \cdot \mathrm{tr}_{i+j}(\mathbf{x}) \cdot t^{i+\mu}. \quad (204)$$

Finally setting $k = m + \mu - 1$ proves the lemma. ■

Lemma 7: For the two distinct indices k_1 and k_2

$$\tilde{K}_{\mathbf{x}}(k_2, k_1) \cdot \left[\mathrm{tr}_n^{(k_1)}(\mathbf{x}) - \mathrm{tr}_n^{(k_2)}(\mathbf{x})\right] = \mathrm{tr}_{n-1}^{(k_1,k_2)}(\mathbf{x}). \quad (205)$$

Proof: The lemma can be derived via a generating function approach. To this end we show

$$\begin{aligned} \prod_{l_1 \neq k_1}(1 + x_{l_1} \cdot t) - \prod_{l_2 \neq k_2}(1 + x_{l_2} \cdot t) &= \left(\frac{1}{1 + x_{k_1} t} - \frac{1}{1 + x_{k_2} t}\right) \cdot \prod_{l}(1 + x_l \cdot t) \quad (206) \\ &= (x_{k_2} - x_{k_1}) \cdot t \cdot \prod_{l \neq \{k_1,k_2\}} (1 + x_l \cdot t) \end{aligned}$$

Comparing like powers of t proves the lemma. ■

Lemma 8: For $1 \leq k \leq m$ and $0 \leq n \leq m-1$

$$x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}) \cdot \sum_{l=1, l \neq k}^{m} x_l \cdot \tilde{K}_{\mathbf{x}}(k,l) + \sum_{l=1, l \neq k}^{m} x_l^2 \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) \cdot \tilde{K}_{\mathbf{x}}(l,k) = (n+1) \cdot \mathrm{tr}_{n+1}^{(k)}(\mathbf{x}). \quad (207)$$

Proof: From Lemma 4 we obtain

$$(n+1) \cdot \mathrm{tr}_{n+1}^{(k)}(\mathbf{x}) = \sum_{l=1, l \neq k}^{m} x_l \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) + x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}) - (n+1) \cdot x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}). \quad (208)$$

Now using (11) we can write

$$\begin{aligned} (n+1) \cdot \mathrm{tr}_{n+1}^{(k)}(\mathbf{x}) &= \sum_{l=1, l \neq k}^{m} {x_l}^2 \tilde{K}_{\mathbf{x}}(l,k) \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) + \quad (209) \\ & \sum_{l=1, l \neq k}^{m} x_l x_k \tilde{K}_{\mathbf{x}}(k,l) \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) - n \cdot x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}). \end{aligned}$$

Comparing (207) and (209), in order to prove the lemma we have to show that

$$x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}) \cdot \sum_{l=1, l \neq k}^{m} x_l \cdot \tilde{K}_{\mathbf{x}}(k,l) = \sum_{l=1, l \neq k}^{m} x_l x_k \tilde{K}_{\mathbf{x}}(k,l) \cdot \mathrm{tr}_n^{(l)}(\mathbf{x}) - n \cdot x_k \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}). \quad (210)$$

We rewrite (210) as

$$n \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}) = \sum_{l=1, l \neq k}^{m} x_l \cdot \tilde{K}_{\mathbf{x}}(k,l) \cdot \left[\mathrm{tr}_n^{(l)}(\mathbf{x}) - \mathrm{tr}_n^{(k)}(\mathbf{x})\right]. \tag{211}$$

Now using Lemma 7 we get

$$n \cdot \mathrm{tr}_n^{(k)}(\mathbf{x}) = \sum_{l=1, l \neq k}^{m} x_l \cdot \mathrm{tr}_{n-1}^{(l,k)}(\mathbf{x}) \tag{212}$$

and finally by Lemma 3 we can prove the lemma. ■

APPENDIX VI
EXPANSIONS IN PARTIAL FRACTIONS

The two lemmas of this section are given without proof.

Lemma 9: For integer k and $0 \leq k \leq m$

$$\frac{(-1)^k \cdot t^k}{\prod_{l=1}^{m}(1 + x_l \cdot t)} = \sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot \frac{x_l^{m-k-1}}{1 + x_l \cdot t}. \tag{213}$$

Lemma 10: For integer k and $0 \leq k \leq m$

$$\frac{(-1)^k}{t^k \cdot \prod_{l=1}^{m}(1 + x_l \cdot t)} = \sum_{l=1}^{m} K_{\mathbf{x}}(l) \cdot \frac{x_l^{m+k-1}}{1 + x_l \cdot t} + \sum_{j=1}^{k} (-1)^j \cdot \frac{\mathrm{h}_{k-j}(\mathbf{x})}{t^j}. \tag{214}$$

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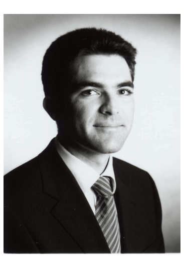

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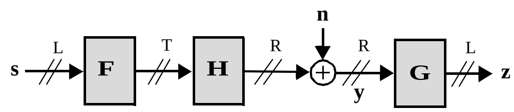

Fig. 1. System Model

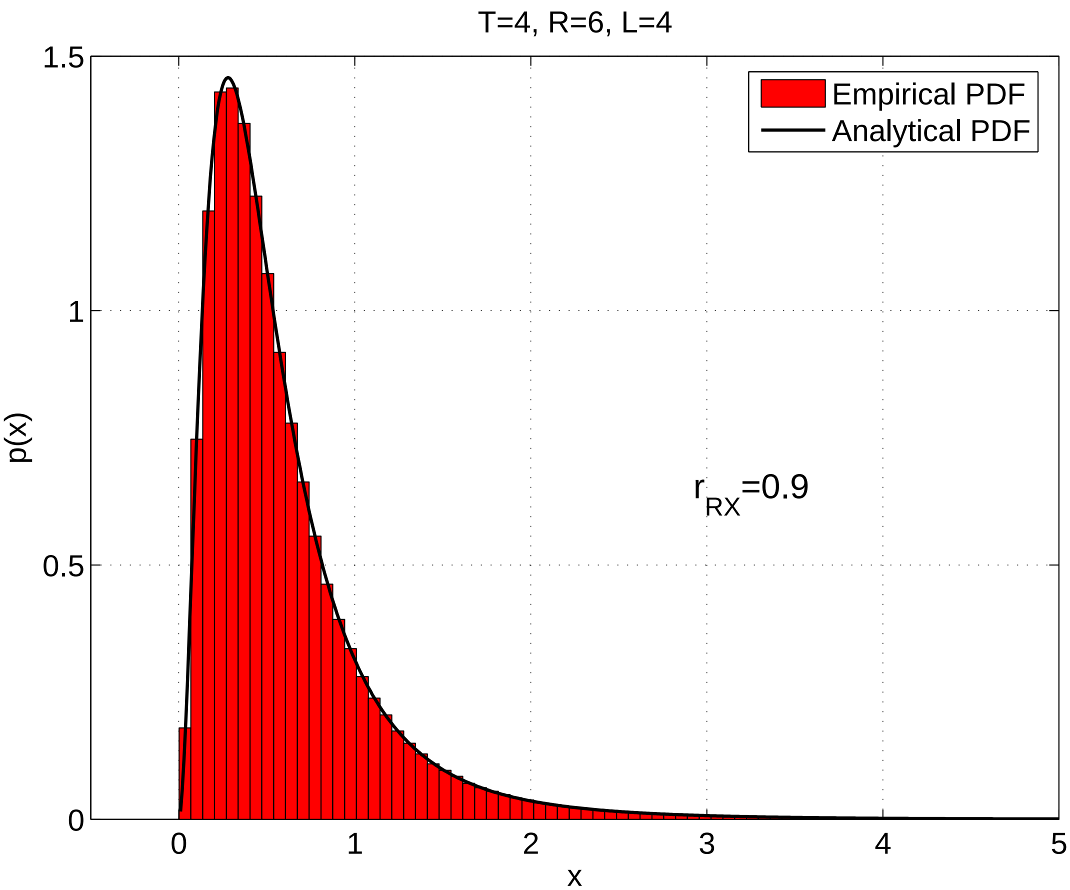

Fig. 2. Probability Density Function, $r_{RX} = 0.9$, $T = L = 4$, $R = 6$

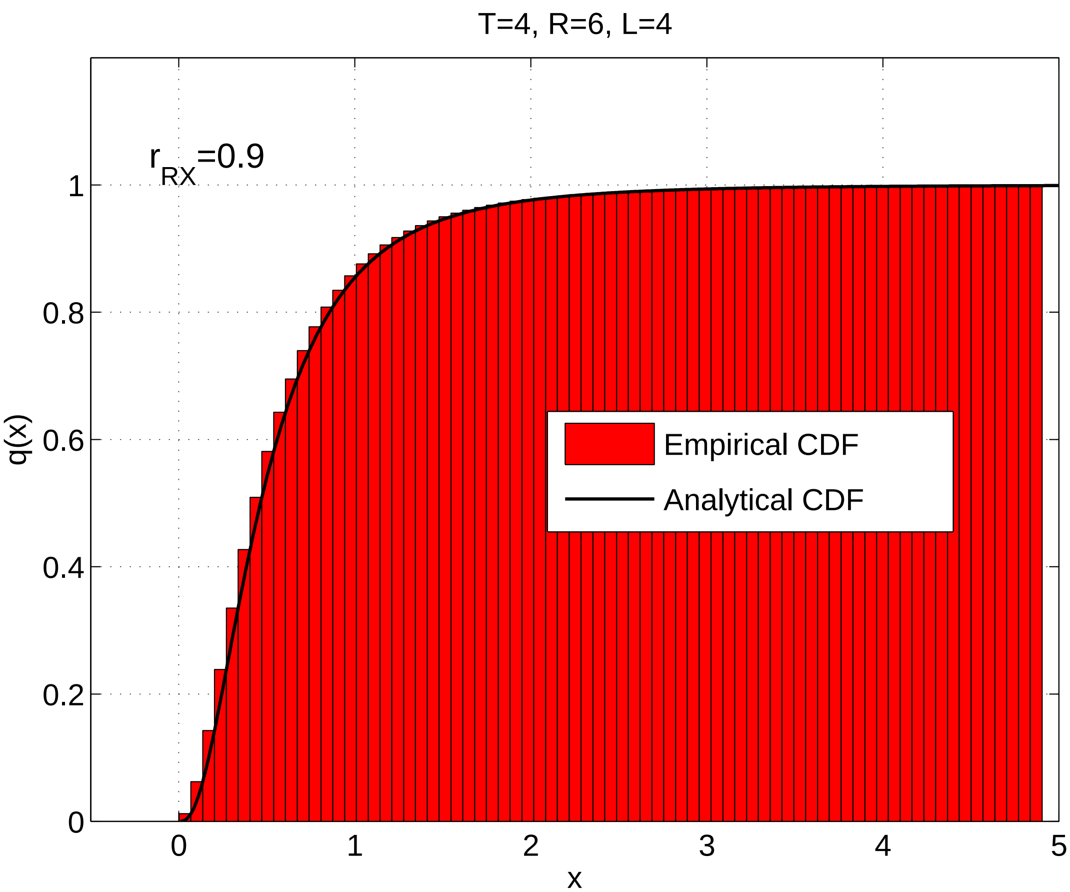

Fig. 3. Cumulative Distribution Function, $r_{RX} = 0.9$, $T = L = 4$, $R = 6$

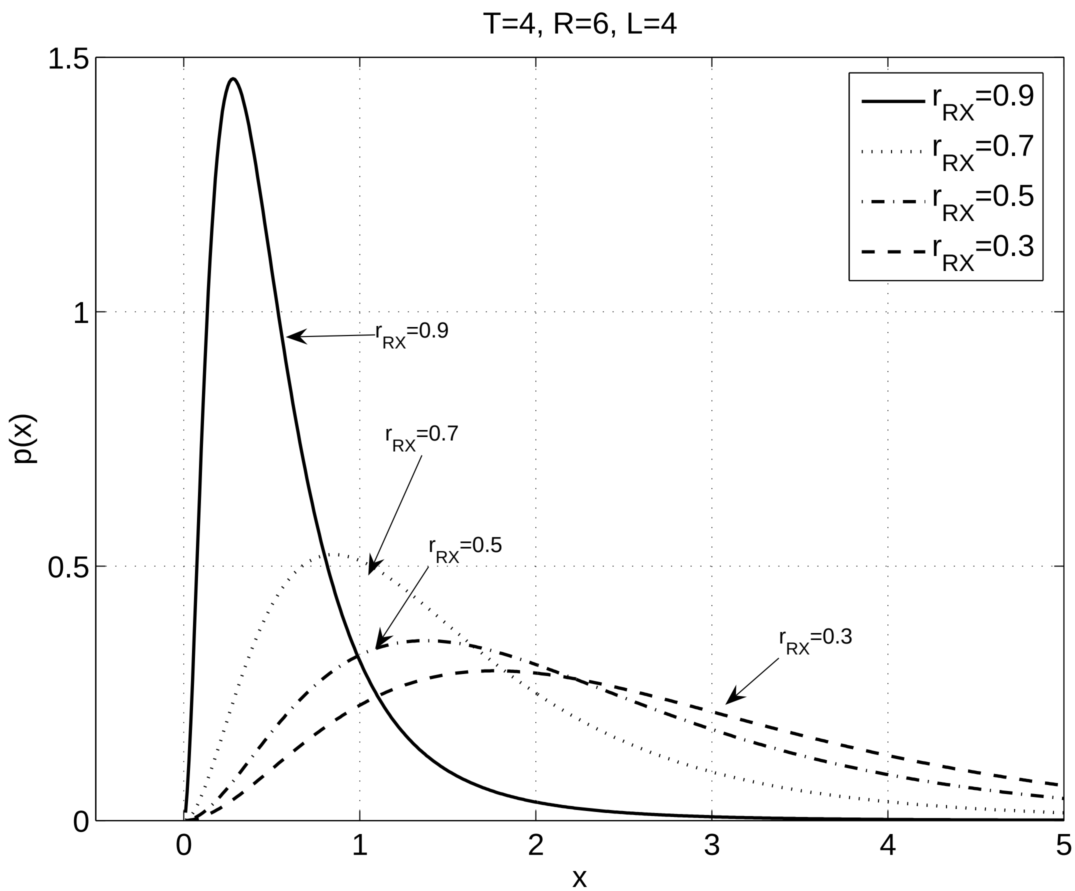

Fig. 4. Probability Distribution Function, $r_{RX} = \{0.3, 0.5, 0.7, 0.9\}$, $T = L = 4$, $R = 6$

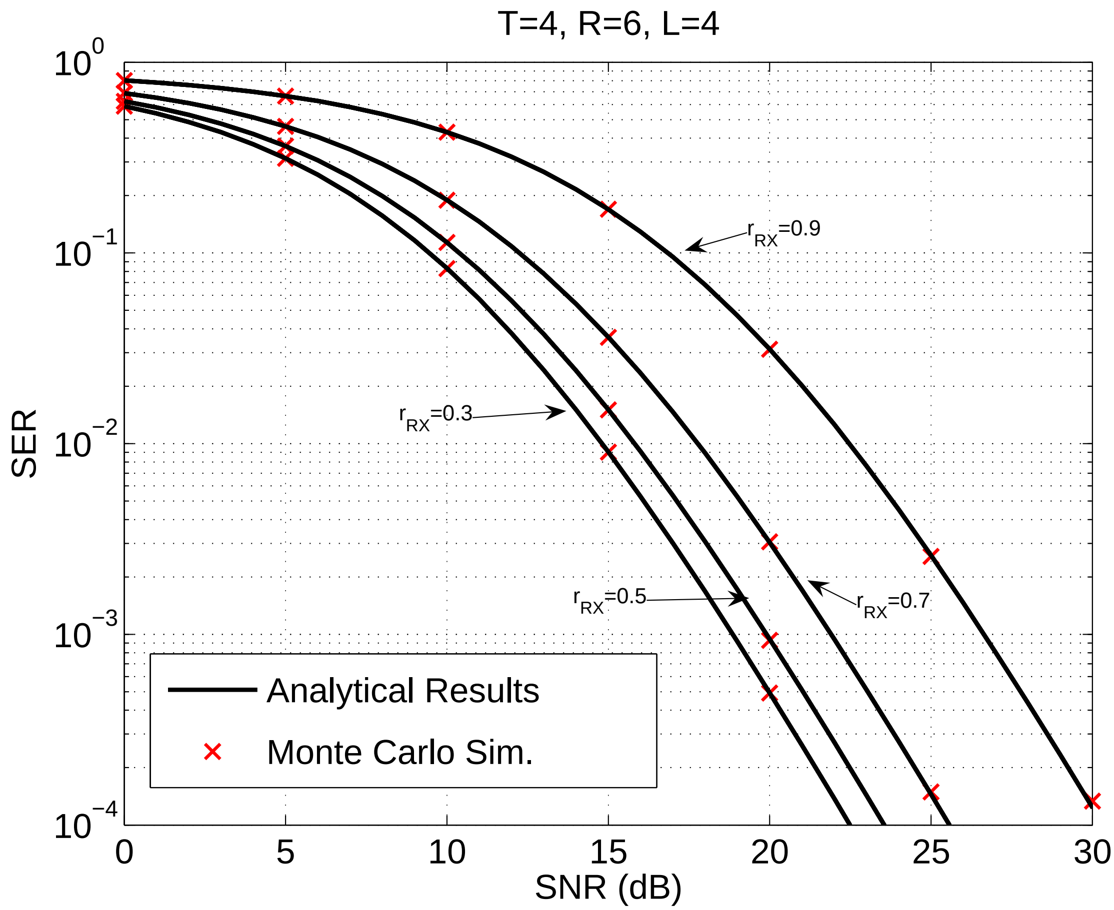

Fig. 5. Symbol Error Rate, $r_{RX} = \{0.3, 0.5, 0.7, 0.9\}$, $T = L = 4$, $R = 6$, $M = 16$ QAM

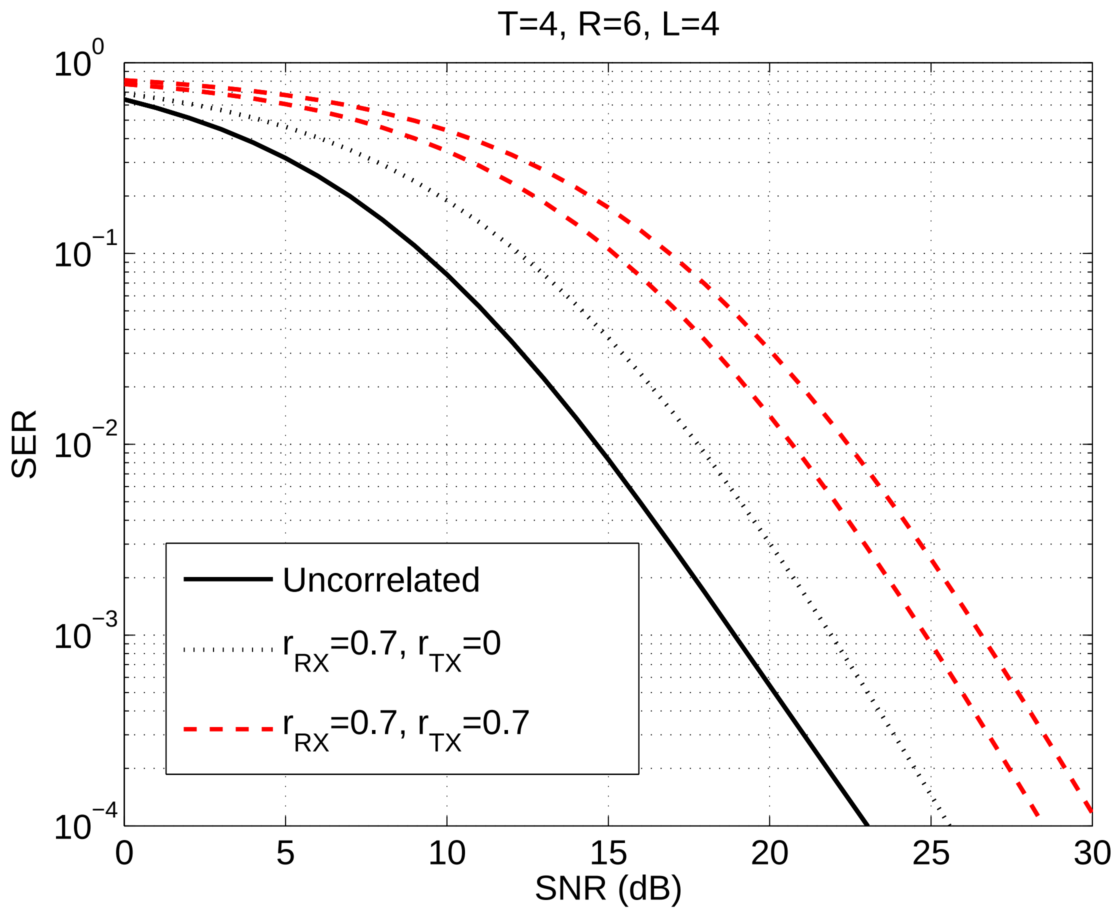

Fig. 6. Symbol Error Rate, $T = L = 4$, $R = 6$, $M = 16$ QAM

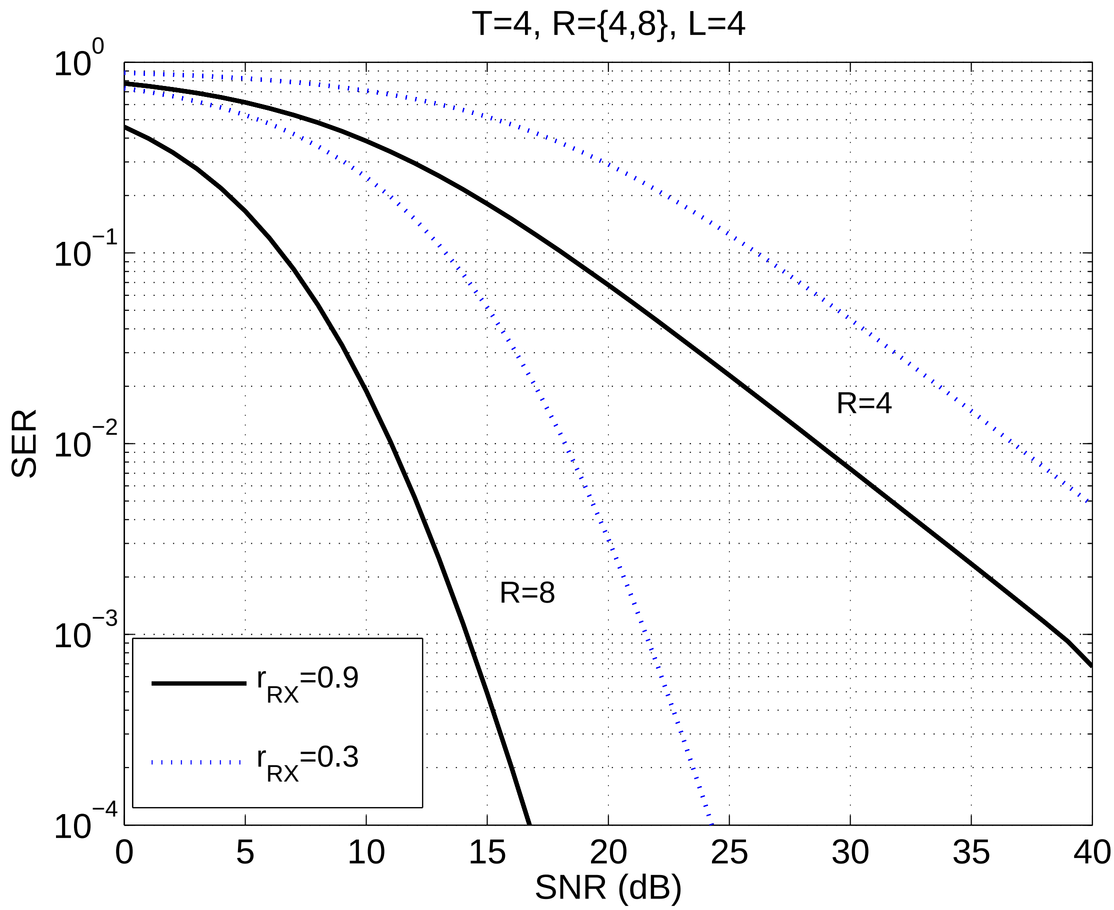

Fig. 7. Symbol Error Rate, $r_{RX} = \{0.3, 0.9\}$, $T = L = 4$, $R = \{4, 8\}$, $M = 16$

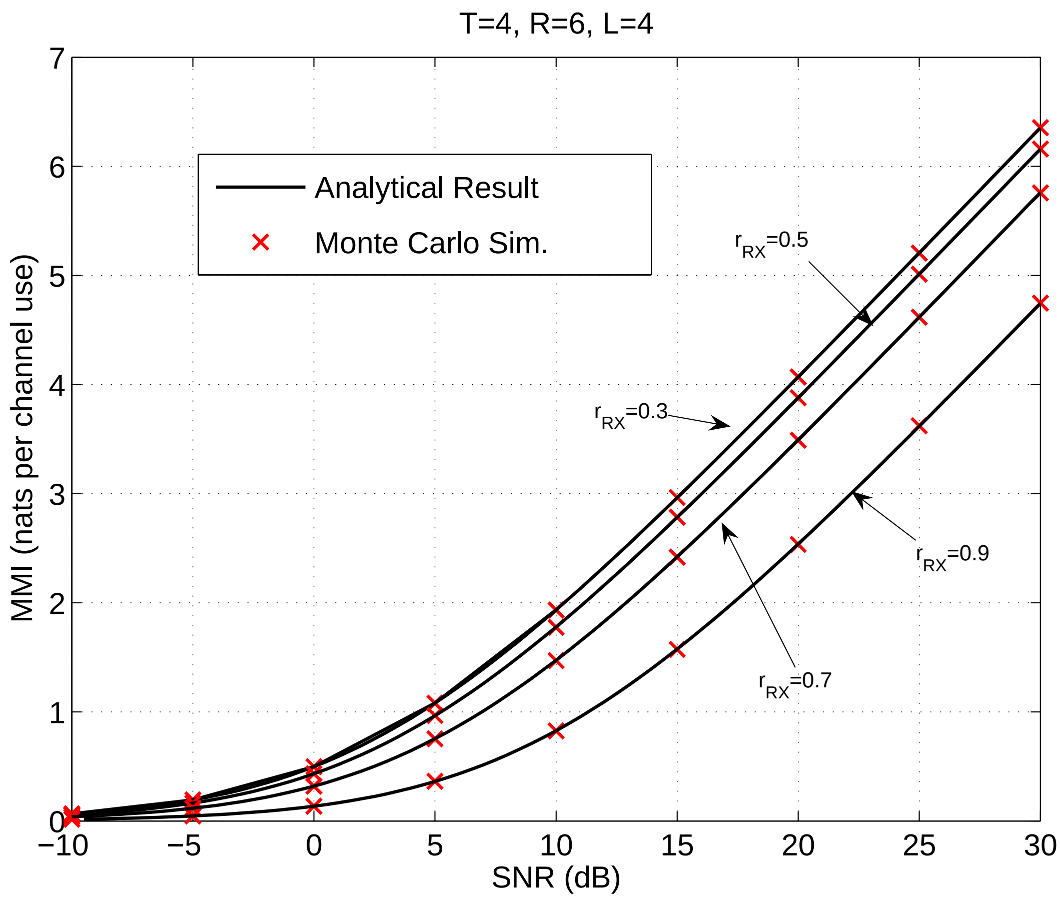

Fig. 8. Mean Mutual Information, $r_{RX} = \{0.3, 0.5, 0.7, 0.9\}$, $T = L = 4$, $R = 6$